Topological nature of the Fu-Kane-Mele invariants

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Outline

1. Time reversal symmetries and "Quaternionic" structures

2. The role of the (involutive) base space

3. In the search of a classifying object

4. FKMM vs. Fu-Kane-Mele

Topological Quantum Systems with odd TRS's

*Let $\mathbb{B}$ a topological space, (**"Brillouin zone"**). Assume that:*

- *$\mathbb{B}$ is a CW-complex (compact, Hausdorff and path-connected);*

DEFINITION (*Topological Quantum System (TQS)*)

*Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{K}(\mathcal{H})$ the algebra of compact operators. A **TQS** is a self-adjoint map*

$$\mathbb{B} \ni k \longmapsto H(k) = H(k)^* \in \mathcal{K}(\mathcal{H})$$

***continuous** with respect to the norm-topology.*

☞ *The **spectrum** $\sigma(H(k)) = \{E_j(k) \mid j \in \mathcal{I} \subseteq \mathbb{Z}\} \subset \mathbb{R}$, is a sequence of eigenvalues ordered according to*

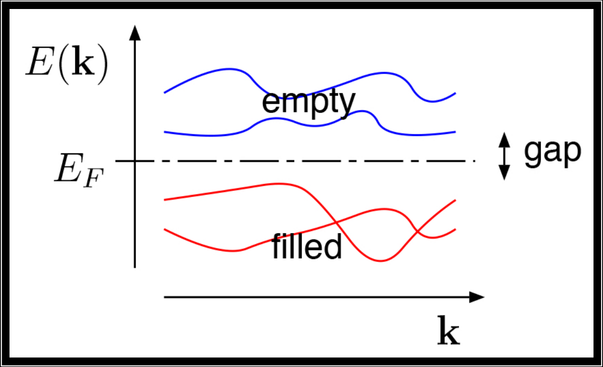
$$\ldots E_{-2}(k) \leqslant E_{-1}(k) < 0 \leqslant E_1(k) \leqslant E_2(k) \leqslant \ldots$$

☞ *The maps $k \mapsto E_j(k)$ are **continuous** (energy bands) ...*

Topological Quantum Systems with odd TRS's

... namely a band spectrum

$$H(k)\,\psi_j(k) \;=\; E_j(k)\,\psi_j(k)\,, \qquad\qquad k \in \mathbb{B}$$



☞ *Usually an energy* ***gap*** *separates the filled* ***valence*** *bands from the empty* ***conduction*** *bands. The* ***Fermi level*** E_F *characterizes the gap.*

Topological Quantum Systems with odd TRS's

A homeomorphism $\tau : \mathbb{B} \to \mathbb{B}$ is called ***involution*** *if $\tau^2 = \mathrm{Id}_{\mathbb{B}}$. The pair $(\mathbb{B}, \tau)$ is called an* ***involutive space*** *and $\mathbb{B}^\tau \subset \mathbb{B}$ is the subsetset of* ***invariant points****. Each space $\mathbb{B}$ admits (at least) the* ***trivial involution*** *$\tau_{\mathrm{triv}} := \mathrm{Id}_{\mathbb{B}}$.*

DEFINITION (*TQS with time-reversal symmetry*)

Let $(\mathbb{B}, \tau)$ be an involutive space, $\mathcal{H}$ a separable Hilbert space endowed with a ***complex conjugation C****. A TQS $\mathbb{B} \ni k \mapsto H(k)$ has a* ***time-reversal symmetry*** *(TRS) of parity $\eta \in \{\pm 1\}$ if there is a continuous unitary-valued map $k \mapsto U(k)$ such that*

$$U(k)\; H(k)\; U(k)^* = C\; H(\tau(k))\; C\,, \qquad C\; U(\tau(k))\; C = \eta U(k)^*\;.$$

A TQS with an ***odd*** *TRS (i.e. $\eta = -1$) is called of class* **AII**.

The Serre-Swan construction

- *An* ***isolated family*** *of energy bands is any (finite) collection* $\{E_{j_1}(\cdot), \ldots, E_{j_m}(\cdot)\}$ *of energy bands such that*

$$\min_{k \in \mathbb{B}} \operatorname{dist}\left(\bigcup_{s=1}^{m} \{E_{j_s}(k)\} \,,\, \bigcup_{j \in \mathcal{I} \setminus \{j_1, \ldots, j_m\}} \{E_j(k)\} \right) = C_g > 0\,.$$

  This is usually called ***gap condition***.

- *An isolated family is described by the* ***Fermi projection***

$$P_F(k) := \sum_{s=1}^{m} |\psi_{j_s}(k)\rangle\langle\psi_{j_s}(k)|\,.$$

  This is a ***continuous*** *projection-valued map*

$$\mathbb{B} \ni k \longmapsto P_F(k) \in \mathcal{K}(\mathcal{H})\,.$$

The Serre-Swan construction

- *For each* $k \in \mathbb{B}$

$$\mathcal{H}_k \ := \ \mathrm{Ran}\ P_F(k) \ \subset \ \mathcal{H}$$

is a subspace of $\mathcal{H}$ *of* ***fixed*** *dimension* m*.*

- *The collection*

$$\mathcal{E}_F \ := \ \bigsqcup_{k \in \mathbb{B}} \mathcal{H}_k$$

is a topological space (said ***total space****) and the* ***map***

$$\pi : \ \mathcal{E}_F \ \longrightarrow \ \mathbb{B}$$

defined by $\pi(k, \mathrm{v}) = k$ *is continuous (and open).*

This is a (rank-m*)* ***complex*** *vector bundle called* ***Bloch-bundle****.*

The Serre-Swan construction

☞ *An **odd** TRS induces a **"Quaternionic"** structure on the Bloch-bundle.*

DEFINITION (*Atiyah, 1966 - Dupont, 1969*)

Let $(\mathbb{B}, \tau)$ *be an involutive space and* $\mathscr{E} \to \mathbb{B}$ *a* ***complex*** *vector bundle. Let* $\Theta : \mathscr{E} \to \mathscr{E}$ *an* ***homeomorphism*** *such that*

$$\Theta \ : \ \mathscr{E}|_k \longrightarrow \mathscr{E}|_{\tau(k)} \qquad \text{is } \textbf{anti}\text{-linear} \ .$$

$[\mathcal{R}]$ - *The pair* $(\mathscr{E}, \Theta)$ *is a* ***"Real"****-bundle over* $(\mathbb{B}, \tau)$ *if*

$$\Theta^2 \ : \ \mathscr{E}|_k \xrightarrow{+1} \mathscr{E}|_k \qquad \forall \ k \in \mathbb{B} \ ;$$

$[\mathcal{Q}]$ - *The pair* $(\mathscr{E}, \Theta)$ *is a* ***"Quaternionic"****-bundle over* $(\mathbb{B}, \tau)$ *if*

$$\Theta^2 \ : \ \mathscr{E}|_k \xrightarrow{-1} \mathscr{E}|_k \qquad \forall \ k \in \mathbb{B} \ .$$

The classification problem

DEFINITION (*Topological phases*)

Let $\mathbb{B} \ni k \longmapsto H(k)$ *be an* ***odd TR-symmetric*** *TQS with an isolated family of* m *energy bands and associated* ***"Quaternionic"*** *Bloch bundle* $\mathscr{E}_F \longrightarrow \mathbb{B}$*. The* ***topological phase*** *of the system is specified by*

$$[(\mathscr{E}_F, \Theta)] \ \in \ \mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{B}, \tau) \ .$$

⇓⇓⇓

Main Question:

How to classify $\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{B}, \tau)$ *at least for* ***low-dimensional*** $\mathbb{B}$*?*

The classification problem

Known results for $\dim(\mathbb{B}) \leqslant 3$

- $\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{B}) \overset{c_1}{\simeq} H^2(\mathbb{B}, \mathbb{Z})$ *(Peterson, 1959)*
- $\mathrm{Vec}^m_{\mathcal{R}}(\mathbb{B}, \tau) \overset{c_1^{\mathcal{R}}}{\simeq} H^2_{\mathbb{Z}_2}(\mathbb{B}, \mathbb{Z}(1))$ *(Kahn, 1987 - D. & Gomi, 2014)*

CAZ	TRS	Category	VB
A	0	complex	$\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{B})$
AI	$+$	"Real"	$\mathrm{Vec}^m_{\mathcal{R}}(\mathbb{B}, \tau)$
AII	$-$	"Quaternionic"	$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{B}, \tau)$

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Electrons in a periodic environment

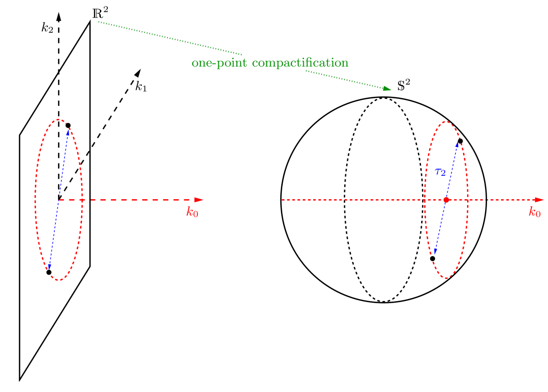
- **Periodic** quantum systems (e.g. absence of **disorder**):
  - $\mathbb{R}^d$-translations $\Rightarrow$ **free** (Dirac) fermions;
  - $\mathbb{Z}^d$-translations $\Rightarrow$ **crystal** (Bloch) fermions.
- The Bloch-Floquet (or Fourier) theory exploits the invariance under translations of a periodic structure to describe the state of the system in terms of the **quasi-momentum** k on the **Brillouin zone** $\mathbb{B}$.
- Complex conjugation (TRS) endows $\mathbb{B}$ with an involution τ.
- Examples are:
  - Gapped electronic systems,
  - BdG superconductors,
  - **Photonic crystals** (M. Lein talk).

Continuous case $\mathbb{B} \equiv \mathbb{S}^{1,d}$

$$\mathbb{S}^d \xrightarrow{\theta_{1,d}} \mathbb{S}^d$$

$$(+k_0, +k_1, \ldots, +k_d) \stackrel{\theta_{1,d}}{\longmapsto} (+k_0, -k_1, \ldots, -k_d)$$

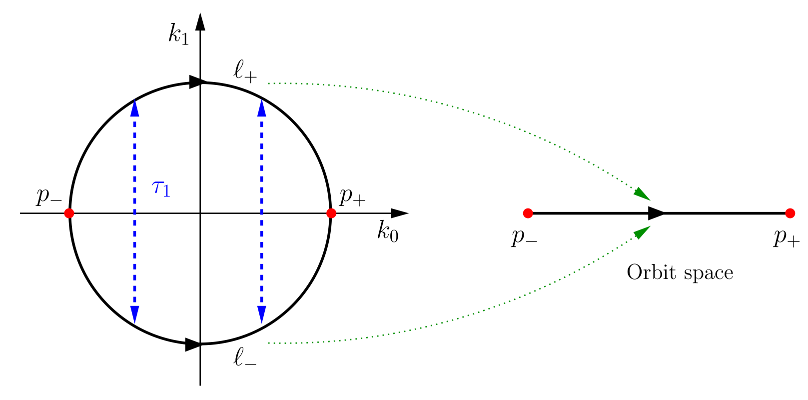
$$\mathbb{S}^{1,d} := (\mathbb{S}^d, \theta_{1,d})$$



Periodic case $\mathbb{B} \equiv \mathbb{T}^{0,d,0}$

$$\mathbb{S}^{1,1} \times \ldots \times \mathbb{S}^{1,1} \xrightarrow{\tau_d := \theta_{1,1} \times \ldots \times \theta_{1,1}} \mathbb{S}^{1,1} \times \ldots \times \mathbb{S}^{1,1}$$

$$\mathbb{T}^{0,d,0} := \underbrace{\mathbb{S}^{1,1} \times \ldots \times \mathbb{S}^{1,1}}_{d\text{ - times}} = (\mathbb{T}^d, \tau_d)$$



Topological states for Bloch electrons

	$d=1$	$d=2$	$d=3$	$d=4$	
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{1,d})$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	*Free*
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{0,d,0})$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^{10} \oplus \mathbb{Z}$	*Periodic*

☞ *First proof for* $d=1,2$ *due to* ***Fu***, ***Kane*** *and* ***Mele (2005 - 2007)*** *based on the*

$$\textbf{\textit{Fu-Kane-Mele index}} := \prod_{k_i \in \mathbb{B}^\tau} \frac{\sqrt{\det[W(k_i)]}}{\mathrm{Pf}[W(k_i)]} \; .$$

Here $\mathbb{B}^\tau \ni k \mapsto W(k)$ *is an antisymmetric matrix built from the Bloch functions.*

! ! It makes sense ***only*** *when* $\mathbb{B}^\tau$ *is* ***finite*** ! !

Topological states for Bloch electrons

An afterwards ...

- ☞ Computed by **Kitaev (2009)** for all **d** by ***K-theory*** (stable range).
- ☞ "Handmade" ***frame construction*** for the case $\mathbb{T}^{0,2,0}$ by **Graf** and **Porta (2013)** and for the case $\mathbb{T}^{0,3,0}$ by **Fiorenza**, **Monaco** and **Panati (2016)** and **Cornean**, **Monaco** and **Teufel (2016)**.
- ☞ **D.** and **Schulz-Baldes (2015)** with ***spectral flux*** (disorder).
- ☞ **Kennedy** and **Zirnbauer (2015)** by the calculation of the ***equivariant homotopy*** (very general but **hard** to compute).
- ☞ **D.** and **Gomi (2015)** by the introduction of the ***FKMM-invariant*** (a characteristic class) and the computation of the **equivariant cohomology** (very general and **not so hard** to compute).

Why more general involutive spaces?

- ☞ ***The external triggering:***
  $\mathbb{B}$ *can be interpreted as the space of* ***control parameters*** *for a quantum system* ***adiabatically perturbed****.*

- ☞ ***The Born-Oppenheimer approximation:***
  Many systems depend by ***slow*** *and* ***fast*** *degrees of freedom (e.g. the Molecular Dynamics). Under certain conditions the slow and fast variables* ***decouple adiabatically*** *(i.e. the fast variables* ***adjust instantly*** *to changes of the slow variables). As a consequence, the fast dynamics is described by an* ***effective Hamiltonian*** *which depends by the slow (****classical****) degrees of freedom. "De facto" one is in a situation described by a* ***TQS***

$$X \ni (q,p) \longmapsto H_{\text{fast}}(q,p)$$

with X *the* ***classical phase space****. The* ***TR symmetry*** *acts on the classical variables and induces an involution on the space* X*.*

Why more general involutive spaces?

- *Therefore* $(\mathbb{B},\tau)$ *can be very general. In particular the* ***fixed-point set*** $\mathbb{B}^\tau$ *could be empty (**free** action) or a sub-manifold of whatever co-dimension (and not necessary a discrete set of points).*

- *For instance there are family of involutive spheres* $\mathbb{S}^{p,q} := (\mathbb{S}^{p+q-1}, \theta_{p,q})$ *with* $\theta_{p,q}$ *defined by*

$$(k_0, k_1, \ldots, k_{p-1}, k_p, \ldots, k_{p+q-1}) \stackrel{\theta_{p,q}}{\mapsto} (k_0, k_1, \ldots, k_{p-1}, -k_p, \ldots, -k_{p+q-1})$$

and of involutive tori

$$\mathbb{T}^{a,b,c} := \underbrace{\mathbb{S}^{2,0} \times \ldots \times \mathbb{S}^{2,0}}_{a-\text{times}} \times \underbrace{\mathbb{S}^{1,1} \times \ldots \times \mathbb{S}^{1,1}}_{b-\text{times}} \times \underbrace{\mathbb{S}^{0,2} \times \ldots \times \mathbb{S}^{0,2}}_{c-\text{times}}$$

- *Recently* ***Gat*** *and* ***Robbins (arXiv:1511.08994)*** *considered the cases* $\mathbb{B} = \mathbb{S}^{0,3}$ *(**rigid rotor**) and* $\mathbb{B} = \mathbb{T}^{1,1,0}$ *(**phase space of slow dynamic of a 1D periodic particle**). In the first case* $\mathbb{B}^\tau = \emptyset$ *and in the second* $\mathbb{B}^\tau = \mathbb{S}^1 \sqcup \mathbb{S}^1$.

Why more general involutive spaces?

! ! Many of the previous approaches just fail when $\mathbb{B}^\tau$ is ***not*** *a finite set* ***! !***

$\Downarrow\Downarrow\Downarrow$

Which object replaces the ***Fu-Kane-Mele index*** *when $\mathbb{B}^\tau$ is not a finite set* ***?***

$\Downarrow\Downarrow\Downarrow$

A characteristic (cohomological) class called ***FKMM-invariant****.*

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Relative equivariant cohomology

The Borel's construction

- (X,τ) *any involutive space and* $(\mathbb{S}^\infty,\theta)$ *the* ***infinite*** *sphere (contractible space) with the* ***antipodal*** *(free) involution:*

$$X_{\sim\tau} := \frac{\mathbb{S}^\infty \times X}{\theta \times \tau} \qquad (\textit{homotopy quotient}) .$$

- $\mathcal{Z}$ *any abelian ring (module, system of coefficients, ...)*

$$H^j_{\mathbb{Z}_2}(X,\mathcal{Z}) := H^j(X_{\sim\tau},\mathcal{Z}) \qquad (\textit{eq. cohomology groups}) .$$

- $\mathbb{Z}(m)$ *the* ***$\mathbb{Z}_2$-local system*** *on* X *based on the module* $\mathbb{Z}$

$$\mathbb{Z}(m) \simeq X\times\mathbb{Z} \qquad \textit{endowed with} \qquad (x,\ell) \mapsto (\tau(x),(-1)^m\ell) .$$

Relative equivariant cohomology

- $H^\bullet_{\mathbb{Z}_2}$ is a (generalized) **cohomology theory** which can be extended to pairs of spaces $Y \subseteq X$ in order to define **relative cohomology groups** $H^\bullet_{\mathbb{Z}_2}(X|Y, \mathcal{Z})$.
- In **[D. - Gomi, 2015]** we showed that

$$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{0,d,0}) \qquad \textit{and} \qquad \mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{1,d-1})\ , \quad d \leqslant 4$$

can be classified by a **characteristic class** with values in $H^2_{\mathbb{Z}_2}(\mathbb{B}|\mathbb{B}^\tau, \mathbb{Z}(1))$: the **FKMM-invariant**.

$$\begin{array}{ccccccc} H^1_{\mathbb{Z}_2}(\mathbb{B}^\tau, \mathbb{Z}(1)) & \xrightarrow{\delta_1} & H^2_{\mathbb{Z}_2}(\mathbb{B}|\mathbb{B}^\tau, \mathbb{Z}(1)) & \xrightarrow{\delta_2} & H^2_{\mathbb{Z}_2}(\mathbb{B}, \mathbb{Z}(1)) & \xrightarrow{r} & H^2_{\mathbb{Z}_2}(\mathbb{B}^\tau, \mathbb{Z}(1)) \\ \wr\| & & & & \wr\| & & \wr\| \\ [\mathbb{B}^\tau, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} & & & & \mathrm{Pic}_{\mathcal{R}}(\mathbb{B}, \tau) & & \mathrm{Pic}_{\mathbb{R}}(\mathbb{B}^\tau) \end{array}$$

☞ *The results in* **[D. - Gomi, 2015]** *only apply to the case of a $\mathbb{B}^\tau$ finite. To consider more general involutive spaces we need more generality !*

The (generalized) FKMM-invariant

THEOREM (*D. - Gomi, 2016 | Part I*)

Given $(\mathbb{B}, \tau)$ *let*

$$\mathrm{Pic}_{\mathcal{R}}\left(\mathbb{B}|\mathbb{B}^{\tau}, \tau\right) := \left\{[(\mathscr{L}, s)] \mid \mathscr{L} \in \mathrm{Pic}_{\mathcal{R}}(\mathbb{B}, \tau),\ s : \mathscr{L}|_{\mathbb{B}^{\tau}} \to \mathbb{U}(1)\right\}$$

*with group structure given by the **tensor product**. Then*

$$\mathrm{Pic}_{\mathcal{R}}\left(\mathbb{B}|\mathbb{B}^{\tau}, \tau\right) \overset{\tilde{\kappa}}{\simeq} H^2_{\mathbb{Z}_2}\left(\mathbb{B}|\mathbb{B}^{\tau}, \mathbb{Z}(1)\right) .$$

This result extends the ***Kahn's isomorphism***

$$\mathrm{Vec}^m_{\mathcal{R}}(\mathbb{B}, \tau) \simeq H^2_{\mathbb{Z}_2}(\mathbb{B}, \mathbb{Z}(1))$$

and indeed can be proved in a similar way.

The (generalized) FKMM-invariant

THEOREM (*D. - Gomi, 2016 | Part II*)

There is a group homomorphism

$$\kappa \ : \ \mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{B},\tau) \ \longrightarrow \ H^2_{\mathbb{Z}_2}\big(\mathbb{B}|\mathbb{B}^\tau, \mathbb{Z}(1)\big)$$

called the ***FKMM-invariant***.

(1) Determinant functor:

If $(\mathscr{E}, \Theta) \in \mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{B},\tau)$ *then* $(\det\mathscr{E}, \det\Theta) \in \mathrm{Pic}_{\mathcal{R}}(\mathbb{B},\tau)$;

(2) Canonical section:

It exists a ***unique*** *(canonical) trivialization*

$$h_{\rm can} \ : \ \det\mathscr{E}|_{\mathbb{B}^\tau} \ \to \ \mathbb{B}^\tau \times \mathbb{C}$$

which define $s_{\rm can}(k) := {h_{\rm can}}^{-1}(k,1)$ *for all* $k \in \mathbb{B}^\tau$.

(3) The mapping $\mathscr{E} \mapsto (\det\mathscr{E}, s_{\rm can}) \in \mathrm{Pic}_{\mathcal{R}}\big(\mathbb{B}|\mathbb{B}^\tau,\tau\big)$:

$$\kappa(\mathscr{E},\Theta) \ := \ \tilde{\kappa}(\det\mathscr{E}, s_{\mathscr{E}}) \ .$$

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Properties of $\kappa : \mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{B},\tau) \longrightarrow H^2_{\mathbb{Z}_2}(\mathbb{B}|\mathbb{B}^\tau, \mathbb{Z}(1))$

(i) ***Isomorphic*** *$\mathcal{Q}$-bundles have the same FKMM-invariant;*

(ii) *If $(\mathscr{E}, \Theta)$ is $\mathcal{Q}$-**trivial** then $\kappa(\mathscr{E}, \Theta) = 0$;*

(iii) *κ is **natural** under the pullback induced by equivariant maps;*

(iv) $\kappa(\mathscr{E}_1 \oplus \mathscr{E}_2, \Theta_1 \oplus \Theta_2) = \kappa(\mathscr{E}_1, \Theta_1) + \kappa(\mathscr{E}_2, \Theta_2)$;

(v) *κ is the image of a **universal class** $\mathfrak{h}_{univ}$;*

(vi) *When $\mathbb{B}^\tau = \{$**finite** collection of points$\}$ then*

$$\kappa(\mathscr{E}, \Theta) \simeq \textit{Fu-Kane-Mele invariants};$$

(vii) *When $\mathbb{B}^\tau = \emptyset$*

$$\kappa(\mathscr{E}, \Theta) \simeq c_1^{\mathcal{R}}(\det\mathscr{E}, \det\Theta);$$

(viii) *When $\mathbb{B}^\tau = \emptyset$ and $\mathrm{Pic}_{\mathcal{Q}}(\mathbb{B},\tau) \neq \emptyset$ then $\mathrm{Pic}_{\mathcal{Q}}(\mathbb{B},\tau)$ is a **torsor** over $\mathrm{Pic}_{\mathcal{R}}(\mathbb{B},\tau)$. Hence*

$$\mathrm{Pic}_{\mathcal{Q}}(\mathbb{B},\tau) \simeq \mathrm{Pic}_{\mathcal{R}}(\mathbb{B},\tau) \simeq H^2_{\mathbb{Z}_2}(\mathbb{B}, \mathbb{Z}(1)).$$

Low dimension ($\dim(\mathbb{B}) \leqslant 3$)

(i) *If* $\dim(\mathbb{B}) \leqslant 2$ *and* $\mathbb{B}^\tau \neq \emptyset$ *then*

$$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{B}, \tau) \;\simeq\; H^2_{\mathbb{Z}_2}\big(\mathbb{B}|\mathbb{B}^\tau, \mathbb{Z}(1)\big)\;.$$

(ii) *If* $\dim(\mathbb{B}) = 3$ *and* $\mathbb{B}^\tau \neq \emptyset$ *the map*

$$\kappa \;:\; \mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{B}, \tau) \;\hookrightarrow\; H^2_{\mathbb{Z}_2}\big(\mathbb{B}|\mathbb{B}^\tau, \mathbb{Z}(1)\big)$$

is ***only injective*** *(even though in many situations like* $\mathbb{S}^{p,q}$ *and* $\mathbb{T}^{a,b,c}$ κ *turns out to be* ***bijective****).*

(iii) *If* $\dim(\mathbb{B}) \leqslant 3$ *and* $\mathbb{B}^\tau = \emptyset$ *then*

$$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{B}, \tau) \;\simeq H^2_{\mathbb{Z}_2}\big(\mathbb{B}, \mathbb{Z}(1)\big)$$

and

$$\mathrm{Vec}^{2m+1}_{\mathcal{Q}}(\mathbb{B}, \tau) \;\simeq \begin{cases} H^2_{\mathbb{Z}_2}\big(\mathbb{B}, \mathbb{Z}(1)\big) & \text{if } \;\mathrm{Pic}_{\mathcal{Q}}(\mathbb{B}, \tau) \neq \emptyset \\ \emptyset & \text{if } \;\mathrm{Pic}_{\mathcal{Q}}(\mathbb{B}, \tau) = \emptyset. \end{cases}$$

Application to involutive spheres $\mathbb{S}^{p,q} := (\mathbb{S}^{p+q-1}, \theta_{p,q})$

$p+q \leqslant 4$	$q=0$	$q=1$	$q=2$	$q=3$	$q=4$
$\mathrm{Vec}_{\mathcal{Q}}^{2m+1}(\mathbb{S}^{0,q})$	$\emptyset$	?	?	$2\mathbb{Z}+1$	?
$\mathrm{Vec}_{\mathcal{Q}}^{2m}(\mathbb{S}^{0,q})$	$\emptyset$	?	?	$2\mathbb{Z}$	?
$\mathrm{Vec}_{\mathcal{Q}}^{2m}(\mathbb{S}^{1,q})$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...
$\mathrm{Vec}_{\mathcal{Q}}^{2m}(\mathbb{S}^{2,q})$	0	?	?	...	
$\mathrm{Vec}_{\mathcal{Q}}^{2m}(\mathbb{S}^{3,q})$	0	?	...		
$\mathrm{Vec}_{\mathcal{Q}}^{2m}(\mathbb{S}^{4,q})$	0	...			

Application to involutive spheres $\mathbb{S}^{p,q} := (\mathbb{S}^{p+q-1}, \theta_{p,q})$

$p+q \leqslant 4$	$q=0$	$q=1$	$q=2$	$q=3$	$q=4$
$\mathrm{Vec}^{2m+1}_{\mathcal{Q}}(\mathbb{S}^{0,q})$	$\emptyset$	0	0	$2\mathbb{Z}+1$	$\emptyset$
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{0,q})$	$\emptyset$	0	0	$2\mathbb{Z}$	0
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{1,q})$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\dots$
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{2,q})$	0	$2\mathbb{Z}$	0	$\dots$	
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{3,q})$	0	0	$\dots$		
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{4,q})$	0	$\dots$			

Application to involutive tori $\mathbb{T}^{a,b,c}$ (fixed points)

$$\mathbb{T}^{a,b,c} := \underbrace{\mathbb{S}^{2,0} \times \ldots \times \mathbb{S}^{2,0}}_{a-\text{times}} \times \underbrace{\mathbb{S}^{1,1} \times \ldots \times \mathbb{S}^{1,1}}_{b-\text{times}} \times \underbrace{\mathbb{S}^{0,2} \times \ldots \times \mathbb{S}^{0,2}}_{c-\text{times}}$$

$a+b \leqslant 3,\ c=0$	$a=0$	$a=1$	$a=2$	$a=3$
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,0,0})$	$\emptyset$	0	0	0
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,1,0})$	0	$2\mathbb{Z}$	?	$\ldots$
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,2,0})$	$\mathbb{Z}_2$	?	$\ldots$	
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,3,0})$	$\mathbb{Z}_2^4$	$\ldots$		

Application to involutive tori $\mathbb{T}^{a,b,c}$ (fixed points)

$$\mathbb{T}^{a,b,c} := \underbrace{\mathbb{S}^{2,0} \times \ldots \times \mathbb{S}^{2,0}}_{a-times} \times \underbrace{\mathbb{S}^{1,1} \times \ldots \times \mathbb{S}^{1,1}}_{b-times} \times \underbrace{\mathbb{S}^{0,2} \times \ldots \times \mathbb{S}^{0,2}}_{c-times}$$

$a+b \leqslant 3,\ c=0$	$a=0$	$a=1$	$a=2$	$a=3$
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,0,0})$	$\emptyset$	0	0	0
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,1,0})$	0	$2\mathbb{Z}$	$(2\mathbb{Z})^2$	$\ldots$
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,2,0})$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus (2\mathbb{Z})^2$	$\ldots$	
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,3,0})$	$\mathbb{Z}_2^4$	$\ldots$		

Application to involutive tori $\mathbb{T}^{a,b,c}$ (free involution)

PROPOSITION (D. - Gomi, 2016)

$$\mathbb{T}^{a,b,c} \simeq \mathbb{T}^{a+c-1,b,1} \qquad \forall\, c \geqslant 2$$

$a+b \leqslant 2,\ c=1$	$a=0$	$a=1$	$a=2$
$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{T}^{a,0,1})$	0	?	?
$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{T}^{a,1,1})$	?	?	...
$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{T}^{a,2,1})$	?	...	

For all $m \in \mathbb{N}$ ***odd*** or ***even***.

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$a+b \leqslant 2,\ c=1$	$a=0$	$a=1$	$a=2$
$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{T}^{a,0,1})$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{T}^{a,1,1})$	$2\mathbb{Z}$	$\mathbb{Z}_2 \oplus (2\mathbb{Z})^2$	$\ldots$
$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{T}^{a,2,1})$	$(2\mathbb{Z})^2$	$\ldots$	

For all $m \in \mathbb{N}$ ***odd*** *or* ***even***.

Thank you for your attention