K-theory of the torus equivariant under the 2-dimensional crystallographic point groups

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The theme of my talk

The equivariant K-theory of the torus acted by

the point group of each 2-dimensional space groups,
or equivalently
the finite subgroups of the mapping class group $GL(2, \mathbb{Z})$.

My talk is based on joint works with
Ken Shiozaki and Masatoshi Sato.

- **Our computational result will be the main theorem.**
- **The computation is motivated by the classification of 3-dimensional topological crystalline insulators.**

The space group

- As is well-known, the group of isometries of $\mathbb{R}^d$ is the semi-direct product of $O(d)$ and $\mathbb{R}^d$:

$$1 \to \mathbb{R}^d \to O(d) \ltimes \mathbb{R}^d \to O(d) \to 1.$$

- A d-dimensional space group (crystallographic group) is a subgroup S,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{R}^d & \longrightarrow & O(d) \ltimes \mathbb{R}^d & \longrightarrow & O(d) & \longrightarrow & 1 \\ & & \cup & & \cup & & \cup & & \\ 1 & \longrightarrow & \Pi & \longrightarrow & S & \longrightarrow & P & \longrightarrow & 1, \end{array}$$

such that

- S contains a rank d lattice $\Pi \cong \mathbb{Z}^d$ of translations,
- the point group $P = S/\Pi$ is a finite subgroup of $O(d)$.

The space group

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{R}^d & \longrightarrow & O(d) \ltimes \mathbb{R}^d & \longrightarrow & O(d) & \longrightarrow & 1 \\ & & \cup & & \cup & & \cup & & \\ 1 & \longrightarrow & \Pi & \longrightarrow & S & \longrightarrow & P & \longrightarrow & 1 \end{array}$$

- S is not necessarily a semi-direct product of P and Π.
  - S is called symmorphic if it is a semi-direct product.
  - S is called nonsymmorphic if not.
- In the nonsymmorphic case, S contains for example a glide, which is a translation along a line ℓ followed by a mirror reflection with respect to ℓ.

The space group

- The space groups are identified if they are conjugate under the affine group $\mathbb{R}^d \ltimes GL^+(d, \mathbb{R})$.
    - $d = 2 \Rightarrow 17$ classes.
    - $d = 3 \Rightarrow 230$ classes.
    - $\cdots$
- In the case of $d = 2$, the space group is also called the plane symmetry group, the wallpaper group, etc.
- To denote the 17 classes of space groups, I will follow:

D. Schattschneider,
The plane symmetry groups: their recognition and their notation.
American Mathematical Monthly 85 (1978), no.6 439–450.

The 2-dimensional space groups (1/2)

label	P	symmorphic?	$P \subset SO(2)$?
p1	1	yes	yes
p2	$\mathbb{Z}_2$	yes	yes
p3	$\mathbb{Z}_3$	yes	yes
p4	$\mathbb{Z}_4$	yes	yes
p6	$\mathbb{Z}_6$	yes	yes

- These point groups are generated by rotations of $\mathbb{R}^2$.
- The other points groups are the dihedral group D_n of degree n and order $2n$:

$$D_n = \langle C, \sigma | \, C^n, \sigma^2, C\sigma C\sigma \rangle.$$

(For example, $D_1 \cong \mathbb{Z}_2$, $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $D_3 = S_3$.)

The 2-dimensional space groups (2/2)

label	P	symmorphic?	$P \subset SO(2)$?
pm	D_1	yes	no
pg	D_1	no	no
cm	D_1	yes	no
pmm	D_2	yes	no
pmg	D_2	no	no
pgg	D_2	no	no
cmm	D_2	yes	no
p3m1	D_3	yes	no
p31m	D_3	yes	no
p4m	D_4	yes	no
p4g	D_4	no	no
p6m	D_6	yes	no

The point group acts on T^2

- Naturally, the point group $P = S/\Pi$ acts on the 2-dimensional torus $T^2 = \mathbb{R}^2/\Pi$.
- By this construction, we get all the 13 classes of finite subgroups in the mapping class group $GL(2, \mathbb{Z})$ of T^2.
- In the case of p1–p6, the cyclic group $\mathbb{Z}_n = \langle C_n | \, C_n^n \rangle$ $(n = 1, 2, 3, 4, 6)$ is embedded into $SL(2, \mathbb{Z})$ through:

$$C_1 = C_6^6 = C_4^4 = 1,$$

$$C_2 = C_6^3 = C_4^2 = -1, \qquad C_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$C_3 = C_6^2, \qquad C_6 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

The point group acts on T^2

- The groups $D_n = \langle C, \sigma |\ C^n, \sigma^2, C\sigma C\sigma \rangle$ $(n = 1, 2, 4)$ are generated by the following matrices in $GL(2, \mathbb{Z})$:

label	P	C	σ
pm/pg	$D_1 = \mathbb{Z}_2$	$C_1 = 1$	σ_x
cm	$D_1 = \mathbb{Z}_2$	$C_1 = 1$	σ_d
pmm/pmg/pgg	$D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$	$C_2 = -1$	σ_x
cmm	$D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$	$C_2 = -1$	σ_d
p4m/p4g	D_4	C_4	σ_x

$$C_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_d = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The point group acts on T^2

- The groups $D_n = \langle C, \sigma \mid C^n, \sigma^2, C\sigma C\sigma \rangle$ $(n = 3, 6)$ are generated by the following matrices in $GL(2, \mathbb{Z})$:

label	P	C	σ
p3m1	D_3	$C_3 = C_6^2$	σ_x
p31m	D_3	$C_3 = C_6^2$	σ_y
p6m	D_6	C_6	σ_y

$$C_6 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Nonsymmorphic space group and twist

- **It happens that a symmorphic group and a nonsymmorphic group share the same point group P.**
- **If a space group is nonsymmorphic, then there is an associated group 2-cocycle with values in the group $C(T^2, U(1))$ of $U(1)$-valued functions, which is regarded as a right module over the point group P.**
- **Such group cocycles provide equivariant twists, namely the data playing roles of local systems for equivariant K-theory, classified by $F^2H^3_P(T^2;\mathbb{Z}) \subset H^3_P(T^2;\mathbb{Z})$. (This classification of twists will be reviewed later.)**

label	P	ori	$F^2 H^3_P(T^2)$	basis
p1	1	$+$	0	
p2	$\mathbb{Z}_2$	$+$	0	
p3	$\mathbb{Z}_3$	$+$	0	
p4	$\mathbb{Z}_4$	$+$	0	
p6	$\mathbb{Z}_6$	$+$	0	
pm/pg	D_1	$-$	$\mathbb{Z}_2$	τ_{pg}
cm	D_1	$-$	0	
pmm/pmg/pgg	D_2	$-$	$\mathbb{Z}_2^{\oplus 3}$	$\tau_{\mathrm{pmg}}, \tau_{\mathrm{pgg}}, c$
cmm	D_2	$-$	$\mathbb{Z}_2$	c
p3m1	D_3	$-$	0	
p31m	D_3	$-$	0	
p4m/p4g	D_4	$-$	$\mathbb{Z}_2^{\oplus 2}$	τ_{p4g}, c
p6m	D_6	$-$	$\mathbb{Z}_2$	c

Main result

- Associated to an action of a finite group P on T^2 and an equivariant twist τ on T^2, we have the P-equivariant τ-twisted K-theory $K_P^{\tau+n}(T^2) \cong K_P^{\tau+n+2}(T^2)$.
- The equivariant twisted K-theory is a module over the representation ring $R(P) = K_P^0(\mathrm{pt})$.

Our main result is the determination of the $R(P)$-module structure of $K_P^{\tau+n}(T^2)$, where

- $n = 0, 1$,
- P ranges the point groups of the 2d space groups,
- τ ranges twists classified by $F^2 H_P^3(T^2; \mathbb{Z})$.

- In the following, the module structure will be omitted.

Theorem [Shiozaki–Sato–G] (1/3)

label	P	τ	$K_P^{\tau+0}(T^2)$	$K_P^{\tau+1}(T^2)$
p1	1	0	$\mathbb{Z}^{\oplus 2}$	$\mathbb{Z}^{\oplus 2}$
p2	$\mathbb{Z}_2$	0	$\mathbb{Z}^{\oplus 6}$	0
p3	$\mathbb{Z}_3$	0	$\mathbb{Z}^{\oplus 8}$	0
p4	$\mathbb{Z}_4$	0	$\mathbb{Z}^{\oplus 9}$	0
p6	$\mathbb{Z}_6$	0	$\mathbb{Z}^{\oplus 10}$	0
pm	D_1	0	$\mathbb{Z}^{\oplus 3}$	$\mathbb{Z}^{\oplus 3}$
pg	D_1	τ_{pg}	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$
cm	D_1	0	$\mathbb{Z}^{\oplus 2}$	$\mathbb{Z}^{\oplus 2}$

Theorem [Shiozaki–Sato–G] (2/3)

label	P	τ	$K_P^{\tau+0}(T^2)$	$K_P^{\tau+1}(T^2)$
pmm	D_2	0	$\mathbb{Z}^{\oplus 9}$	0
pmm	D_2	c	$\mathbb{Z}$	$\mathbb{Z}^{\oplus 4}$
pmg	D_2	$\begin{cases} \tau_{\mathrm{pmg}}, \tau_{\mathrm{pmg}} + c \\ \tau_{\mathrm{pmg}} + \tau_{\mathrm{pgg}}, \\ \tau_{\mathrm{pmg}} + \tau_{\mathrm{pgg}} + c \end{cases}$	$\mathbb{Z}^{\oplus 4}$	$\mathbb{Z}$
pgg	D_2	$\tau_{\mathrm{pgg}}, \tau_{\mathrm{pgg}} + c$	$\mathbb{Z}^{\oplus 3}$	$\mathbb{Z}_2$
cmm	D_2	0	$\mathbb{Z}^{\oplus 6}$	0
cmm	D_2	c	$\mathbb{Z}^{\oplus 2}$	$\mathbb{Z}^{\oplus 2}$

Theorem [Shiozaki–Sato–G] (3/3)

label	P	τ	$K_P^{\tau+0}(T^2)$	$K_P^{\tau+1}(T^2)$
p3m1	D_3	0	$\mathbb{Z}^{\oplus 5}$	$\mathbb{Z}$
p31m	D_3	0	$\mathbb{Z}^{\oplus 5}$	$\mathbb{Z}$
p4m	D_4	0	$\mathbb{Z}^{\oplus 9}$	0
p4m	D_4	c	$\mathbb{Z}^{\oplus 3}$	$\mathbb{Z}^{\oplus 3}$
p4g	D_4	τ_{p4g}	$\mathbb{Z}^{\oplus 6}$	0
p4g	D_4	$\tau_{\mathrm{p4g}} + c$	$\mathbb{Z}^{\oplus 4}$	$\mathbb{Z}$
p6m	D_6	0	$\mathbb{Z}^{\oplus 8}$	0
p6m	D_6	c	$\mathbb{Z}^{\oplus 4}$	$\mathbb{Z}^{\oplus 2}$

Examples of module structures

- I only show the module structures of $K_P^{\tau+n}(T^2)$ in the case of p3m1 and p31m.
- The point group is $D_3 \cong S_3$, and

$$R(D_3) = \mathbb{Z}[A, E]/(1 - A^2, E - AE, 1 + A + E - E^2),$$

where

  - A is the sign representation,
  - E is the unique 2-dimensional irreducible representation.

label	$K^0_{D_3}(T^2)$	$K^1_{D_3}(T^2)$
p3m1	$R(D_3) \oplus (1 + A - E)^{\oplus 2} = \mathbb{Z}^{\oplus 5}$	$(1 - A) = \mathbb{Z}$
p31m	$R(D_3) \oplus (R(D_3)/(E))^{\oplus 2} = \mathbb{Z}^{\oplus 5}$	$(1 - A) = \mathbb{Z}$

Some comments: proceeding works

- **In the papers:**
  1. W. Lück and R. Stamm,
     Computations of K- and L-theory of cocomapct planar groups.
     K-theory 21 249–292, 2000,
  2. M. Yang,
     Crossed products by finite groups acting on low dimensional complexes and applications.
     PhD Thesis, University of Saskatchewan, Saskatoon, 1997.

  the K-theory $K_n(C^*_r(S_\lambda))$ is determined as an abelian group for each 2d space group S_λ, which agrees with our result about $K^{\tau_\lambda+n}_{P_\lambda}(T^2)$.

Some comments: twists

- **There are twists which cannot be realized as group cocycles. The equivariant K-theories twisted by such twists are not completely computed yet.**
- **The role of such a twist in condensed matter seems to be open.**

Some comments : $\mathbb{Z}_2$

- **Recall that there appeared $\mathbb{Z}_2$-summands:**

label	P	τ	$K_P^{\tau+0}(T^2)$	$K_P^{\tau+1}(T^2)$
pg	D_1	τ_{pg}	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$
pgg	D_2	$\tau_{pgg}, \tau_{pgg} + c$	$\mathbb{Z}^{\oplus 3}$	$\mathbb{Z}_2$

- **A consequence is that these $\mathbb{Z}_2$-summands imply a 'new' class of topological insulators which are:**
  - **classified by $\mathbb{Z}_2$,**
  - **realized without the {time-reversal / particle-hole} symmetries.**

  (The well-known topological insulators classified by $\mathbb{Z}_2$ correspond to $KR^{-1}(\mathrm{pt}) = \mathbb{Z}_2$ or $KR^{-2}(\mathrm{pt}) = \mathbb{Z}_2$, and are realized with TRS or PHS.)

- **This is detailed in PRB B91, 155120 (2015).**

1. Main theorem
2. **Gapped system and K-theory**
   $\cdots$ **How twisted K-theory arises?**
3. **Equivariant twist**
   $\cdots$ **a review of twists and their classification**

Gapped system and K-theory

- **Here I would like to explain how a "gapped system with symmetry" leads to an element of K-theory.**
- **If we consider a nonsymmorphic space group as a symmetry, then we get a twisted equivariant K-class.**

Step 1	**:**	**Setting**
Step 2	**:**	**Bloch bundle**
Step 3	**:**	**Symmetry**

Step 1 : Setting

- **Let us consider the following mathematical setting:**
  1. **A lattice $\Pi \subset \Pi \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^d$ of rank d.**
  2. **A symmetric bilinear form $\langle \ , \ \rangle : \Pi \times \Pi \to \mathbb{Z}$ whose induced bilinear form on $\mathbb{R}^d$ is positive definite.**
  3. **A space group S:**

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{R}^d & \longrightarrow & O(d) \ltimes \mathbb{R}^d & \longrightarrow & O(d) & \longrightarrow & 1 \\ & & \cup & & \cup & & \cup & & \\ 1 & \longrightarrow & \Pi & \longrightarrow & S & \xrightarrow{\pi} & P & \longrightarrow & 1, \end{array}$$

     such that P preserves the bilinear form on Π.
  4. **A unitary representation $U : P \to U(V)$ of P on a Hermitian vector space V of finite rank.**

Quantum system with symmetry

- **We then consider the following quantum system on $\mathbb{R}^d$:**
  1. **The quantum Hilbert space: $L^2(\mathbb{R}^d, V)$.**
  2. **The symmetry: $S \curvearrowright L^2(\mathbb{R}^d, V)$.**

$$\psi(x) \overset{g}{\mapsto} (\rho(g)\psi)(x) = U(\pi(g))\psi(g^{-1}x).$$

  3. **The Hamiltonian: a self-adjoint operator H on $L^2(\mathbb{R}^d, V)$ such that $H \circ \rho(g) = \rho(g) \circ H$.**

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  3. **The Hamiltonian: a self-adjoint operator H on $L^2(\mathbb{R}^d, V)$ such that $H \circ \rho(g) = \rho(g) \circ H$.**
- **I will not specify whether H is bounded or not.**
- **But, I will assume H is 'gapped' in the sequel.**
  (This is a property of a topological insulator.)

Step 2: Bloch bundle

- To carry out the 'Bloch transformation', let us denote the Pontryagin dual of Π by $\hat{\Pi} = \mathrm{Hom}(\Pi, U(1))$.
- $\hat{\Pi}$ is identified with the torus $\mathbb{R}^d/\Pi$.

$$k \in \mathbb{R}^d/\Pi \leftrightarrow [m \mapsto e^{2\pi i \langle m,k \rangle}] \in \hat{\Pi}.$$

- Then, define $L^2_\Pi(\hat{\Pi} \times \mathbb{R}^d, V) \subset L^2(\hat{\Pi} \times \mathbb{R}^d, V)$ to be the subspace of L^2-functions $\hat{\psi}(k, x)$ which are quasi-periodic in the second variable:

$$\hat{\psi}(k, x+m) = e^{2\pi i \langle m,k \rangle} \hat{\psi}(k, x) \quad (m \in \Pi)$$

Bloch transformation

$$\hat{\psi}(k,x) \in L^2_{\Pi}(\hat{\Pi} \times \mathbb{R}^d, V) \subset L^2(\hat{\Pi} \times \mathbb{R}^d, V)$$
$$\Leftrightarrow \hat{\psi}(k, x+m) = e^{2\pi i \langle m,k \rangle} \hat{\psi}(k,x) \quad (m \in \Pi)$$

- **The Bloch transformation is defined as follows:**

$$\hat{\mathcal{B}}: \ L^2(\mathbb{R}^d, V) \to L^2_{\Pi}(\hat{\Pi} \times \mathbb{R}^d, V),$$
$$(\hat{\mathcal{B}}\psi)(k,x) = \sum_{n \in \Pi} e^{-2\pi i \langle n,k \rangle} \psi(x+n).$$

- **The following gives the inverse transformation:**

$$\mathcal{B}: \ L^2_{\Pi}(\hat{\Pi} \times \mathbb{R}^d, V) \to L^2(\mathbb{R}^d, V),$$
$$(\mathcal{B}\hat{\psi})(x) = \int_{k \in \hat{\Pi}} \hat{\psi}(k,x) dk.$$

The Poincaré line bundle

- As a result, we get the identification of Hilbert spaces:

$$L^2(\mathbb{R}^d, V) \cong L^2_\Pi(\hat{\Pi} \times \mathbb{R}^d, V).$$

- To get a further identification, let us introduce the Poincaré line bundle $L \to \hat{\Pi} \times \mathbb{R}^d/\Pi$.
- This is the quotient of the product line bundle

$$\hat{\Pi} \times \mathbb{R}^d \times \mathbb{C} \to \hat{\Pi} \times \mathbb{R}^d$$

under the free action of $m \in \Pi$ given by

$$\begin{array}{ccc} \hat{\Pi} \times \mathbb{R}^d \times \mathbb{C} & \xrightarrow{m} & \hat{\Pi} \times \mathbb{R}^d \times \mathbb{C}. \\ (k, x, z) & \mapsto & (k, x + m, e^{2\pi i \langle m, k \rangle} z) \end{array}$$

Further identification

- From $L \to \hat{\Pi} \times \mathbb{R}^d/\Pi$, we can construct an infinite dimensional vector bundle $\mathcal{E} \to \hat{\Pi}$:

$$\mathcal{E} = \bigcup_{k \in \hat{\Pi}} L^2(\mathbb{R}^d/\Pi, L|_{\{k\} \times \mathbb{R}^d/\Pi}).$$

- Then, the Hilbert space $L^2_{\Pi}(\hat{\Pi} \times \mathbb{R}^d, V)$ can be identified with the space of L^2-sections of $\mathcal{E} \otimes V \to \hat{\Pi}$.

$$L^2_{\Pi}(\hat{\Pi} \times \mathbb{R}^d, V) \cong L^2(\hat{\Pi}, \mathcal{E} \otimes V).$$

- In sum, we have the identifications of Hilbert spaces:

$$L^2(\mathbb{R}^d, V) \cong L^2_{\Pi}(\hat{\Pi} \times \mathbb{R}^d, V) \cong L^2(\hat{\Pi}, \mathcal{E} \otimes V).$$

Gapped condition and the Bloch bundle

$$L^2(\mathbb{R}^d, V) \cong L^2_\Pi(\hat{\Pi} \times \mathbb{R}^d, V) \cong L^2(\hat{\Pi}, \mathcal{E} \otimes V).$$

- By the assumption that the Hamiltonian H on $L^2(\mathbb{R}^d, V)$ commutes with the action of $\Pi \subset S$, the induced Hamiltonian $\hat{H}$ on $L^2(\hat{\Pi}, \mathcal{E} \otimes V)$ is induced from a self-adjoint vector bundle map $\hat{\mathcal{H}}$ on $\mathcal{E} \otimes V$.
- As the 'gapped condition', let us assume that a finite number of spectra of $\hat{\mathcal{H}}_k$ on the fiber $\mathcal{E}|_k \otimes V$ is confined in a closed interval as $k \in \hat{\Pi}$ varies.
- Then, by the spectral projection, we get a finite rank vector bundle $E \to \hat{\Pi}$, called the Bloch bundle.
- Its K-class $[E] \in K^0(\hat{\Pi})$ is an invariant of the gapped quantum system we considered.

Step 3 : Symmetry

- Finally, I take the symmetry into account, to make the Bloch bundle into an equivariant vector bundle.
- If the space group is nonsymmorphic, then the resulting equivariant vector bundle will be twisted.
- Let us recall the diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{R}^d & \longrightarrow & O(d) \ltimes \mathbb{R}^d & \longrightarrow & O(d) & \longrightarrow & 1 \\ & & \cup & & \cup & & \cup & & \\ 1 & \longrightarrow & \Pi & \longrightarrow & S & \xrightarrow{\pi} & P & \longrightarrow & 1. \end{array}$$

- Let us define a map $a : P \to \mathbb{R}^d$ by expressing the composition $P \to O(d) \to O(d) \ltimes \mathbb{R}^d$ as $p \mapsto (p, a_p)$.

Group cocycles

- The nonsymmorphic nature of S is measured by the group cocycle with values in the left P-module Π:

$$\nu \in Z^2(P, \Pi)$$

defined by $\nu(p_1, p_2) = a_{p_1} + p_1 a_{p_2} - a_{p_1 p_2}$.

- Then, we get an induced group 2-cocycle with values in the right P-module $C(\hat{\Pi}, U(1))$:

$$\tau \in Z^2(P, C(\hat{\Pi}, U(1)))$$

by $\tau(p_1, p_2; k) = \exp 2\pi i \langle \nu(p_1^{-1}, p_2^{-1}), k\rangle$.

- The cocycle condition for τ:

$$\tau(p_1, p_2 p_3; k)\tau(p_2, p_3; k) = \tau(p_1, p_2 p_3; k)\tau(p_1, p_2; p_3 k).$$

Twisted group action

- For $p \in P$, we define $\rho(p) : L^2(\mathbb{R}^d, V) \to L^2(\mathbb{R}^d, V)$ by

$$\psi(x) \mapsto (\rho(p)\psi)(x) = U(p)\psi(p^{-1}x + a_{p^{-1}}).$$

- Then the map on $L^2(\hat{\Pi}, \mathcal{E} \otimes V)$ corresponding to $\rho(p)$ is identified with the map induced from a map:

$$\begin{array}{ccc} \mathcal{E} \otimes V & \xrightarrow{\rho_{\mathcal{E}\otimes V}(p)} & \mathcal{E} \otimes V \\ \downarrow & & \downarrow \\ \hat{\Pi} & \xrightarrow{p} & \hat{\Pi}, \end{array}$$

where $p : \hat{\Pi} \to \hat{\Pi}$ is $k \mapsto pk$.

- This is a τ-twisted action, in the sense that

$$\rho(p_1; p_2 k)\rho(p_2; k)\xi = \tau(p_1, p_2; k)\rho(p_1 p_2; k)\xi$$

holds for all $\xi \in \mathcal{E}|_k \otimes V$ and $k \in \hat{\Pi}$.

Twisted vector bundle

$$\begin{array}{ccc} \mathcal{E} \otimes V & \xrightarrow{\rho(p)} & \mathcal{E} \otimes V \\ \downarrow & & \downarrow \\ \hat{\Pi} & \xrightarrow{p} & \hat{\Pi}, \end{array}$$

$$\rho(p_1; p_2 k)\rho(p_2; k) = \tau(p_1, p_2; k)\rho(p_1 p_2; k).$$

- **Under the gapped condition, the Bloch bundle $E \subset \mathcal{E} \otimes V$ inherits a τ-twisted action from $\mathcal{E} \otimes V$.**
- **The τ-twisted vector bundle $E \to \hat{\Pi}$ defines a K-class $[E] \in K_P^{\tau+0}(\hat{\Pi})$, which is an invariant of the gapped system with symmetry we considered.**

Some comments

- **A result of Freed–Moore says that: if a finite group G acts on a 'nice' space X and $\tau \in Z^2(G; C(X, U(1)))$, then $K_G^{\tau+0}(X)$ can be realized as the Grothendieck group of the isomorphism classes of finite rank τ-twisted G-equivariant vector bundles on X.**

Some comments

- A result of Freed–Moore says that: if a finite group G acts on a 'nice' space X and $\tau \in Z^2(G; C(X, U(1)))$, then $K_G^{\tau+0}(X)$ can be realized as the Grothendieck group of the isomorphism classes of finite rank τ-twisted G-equivariant vector bundles on X.
- Recall that, in the first setting, we considered a unitary representation $U : P \to U(V)$ of P.
- We can assume this to be a c-projective representation, where $c \in Z^2(P, U(1))$ is the group cocycle with values in the trivial P-module $U(1)$.
- In this case, the resulting Bloch bundle E defines a twisted K-class $[E] \in K_P^{\tau+c+0}(\hat{\Pi})$.

Some comments

- The construction of a twisted vector bundle so far is a version of the 'Macky machine'.
- Let us assume that there is an extension of a finite group P by a finite abelian group Π:

$$1 \to \Pi \to S \xrightarrow{\pi} P \to 1.$$

- A choice of a map $\sigma : P \to S$ such that $\pi \circ \sigma = 1$ defines a 2-cocycle $\tau \in Z^2(P, C(\hat{\Pi}, U(1)))$,

$$\tau(p_1, p_2; \lambda) = \lambda(\sigma(p_1 p_2)^{-1} \sigma(p_1) \sigma(p_2)).$$

Some comments

$$1 \to \Pi \to S \xrightarrow{\pi} P \to 1.$$

Π finite abelian, P finite.

- Then there is an isomorphism of groups:

$$K_S^0(\mathrm{pt}) \cong K_P^{\tau+0}(\hat{\Pi}).$$

$$E = \bigoplus_{\lambda \in \hat{\Pi}} \mathbb{C}_\lambda \otimes \mathrm{Hom}_\Pi(\mathbb{C}_\lambda, E) \mapsto \bigcup_{\lambda \in \hat{\Pi}} \mathrm{Hom}_\Pi(\mathbb{C}_\lambda, E).$$

- In order to justify these machinery in the case where S is a space group, a C^*-algebraic approach is useful, as discussed by G. C. Thiang.

1. **Main theorem**
2. **Gapped system and K-theory**
3. **Equivariant twist**

   ··· **a review of twists and their classification**

   (results from arXiv:1509.09194)

Equivariant twist

- A G-equivariant twist τ (or equivalently a G-equivariant gerbe) on a space X is a datum playing a role of a local system for equivariant K-theory.

$$K_G^n(X) \rightsquigarrow K_G^{\tau+n}(X).$$

- They are classified by the third Borel equivariant cohomology with integer coefficients:

$$[\tau] \in H_G^3(X;\mathbb{Z}).$$

- There are four types of equivariant twists corresponding to the filtration:

$$H_G^3(X) \supset F^1H_G^3(X) \supset F^2H_G^3(X) \supset F^3H_G^3(X)$$

computing the Leray-Serre spectral sequence.

Four types of twists

$$H^3_G(X) \supset F^1H^3_G(X) \supset F^2H^3_G(X) \supset F^3H^3_G(X)$$

- If G fixes at least one point $\mathrm{pt} \in X$, then the four types of twists admit the following interpretation:
    1. Twists realized by group cocycles $\tau \in Z^2(G; U(1))$, classified by $F^3H^3_G(X) = H^3_G(\mathrm{pt})$.
    2. Twists realized by group cocycles $\tau \in Z^2(G; C(X, U(1)))$, classified by $F^2H^3_G(X)$.
    3. Twists realized by central extensions of the groupoid $X//G$ associated to the G-action on X, classified by $F^1H^3_G(X)$.
    4. Twists which cannot be realized by central extensions of $X//G$.

Central extension of $X//G$ with G finite

Definition

A **central extension** (L, τ) of $X//G$ consists of:

- complex line bundles $L_g \to X$, $(g \in G)$
- isomorphisms of line bundles
  $\tau_{g,h}(x) : L_g|_{hx} \otimes L_h|_x \to L_{gh}|_x$, $(g, h \in G)$ making the following diagram commutative:

$$\begin{array}{ccc} L_g|_{hkx} \otimes L_h|_{kx} \otimes L_k|_x & \xrightarrow{1 \otimes \tau_{h,k}(x)} & L_g|_{hkx} \otimes L_{hk}|_x \\ {\scriptstyle \tau_{g,h}(kx)\otimes 1}\downarrow & & \downarrow{\scriptstyle \tau_{g,hk}(x)} \\ L_{gh}|_{kx} \otimes L_k|_x & \xrightarrow{\tau_{gh,k}(x)} & L_{ghk}|_x . \end{array}$$

- A group 2-cocycle $\tau \in Z^2(G; C(X, U(1)))$ is a special example of a central extension such that L_g is trivial.

$$H^3_G(X) \supset F^1H^3_G(X) \supset F^2H^3_G(X) \supset F^3H^3_G(X)$$

- **Because $F^1H^3_G(X) = \mathrm{Ker}[H^3_G(X) \to H^3(X)]$, all the twists on $X = T^2$ can be realized by central extensions.**
- **As is seen, the group cocycles associated to nonsymmorphic space groups and the 'constant' group cocycles are relevant to topological insulators.**
- **As twists, they are classified by $F^2H^3_G(X)$.**
- **Now, there arises mathematically natural questions:**

Questions

- **Let us consider the case where the point group P of a 2d space group acts on T^2.**

Questions

1. **Are there group 2-cocycles other than combinations of group cocycles associated to nonsymmorphic space groups and the 'constant' group cocycles?**
2. **Are there twists (or central extensions) which cannot be realized by group 2-cocycles?**

$$H^3_P(T^2) = F^1H^3_P(T^2) \supset F^2H^3_P(T^2) \supset F^3H^3_P(T^2)$$

Questions

- **Let us consider the case where the point group P of a 2d space group acts on T^2.**

Questions

1. **Are there group 2-cocycles other than combinations of group cocycles associated to nonsymmorphic space groups and the 'constant' group cocycles? No!**
2. **Are there twists (or central extensions) which cannot be realized by group 2-cocycles? Yes!**

$$H^3_P(T^2) = F^1H^3_P(T^2) \supset F^2H^3_P(T^2) \supset F^3H^3_P(T^2)$$

Questions

- Let us consider the case where the point group P of a 2d space group acts on T^2.

Questions

1. Are there group 2-cocycles other than combinations of group cocycles associated to nonsymmorphic space groups and the 'constant' group cocycles? **No!**
2. Are there twists (or central extensions) which cannot be realized by group 2-cocycles? **Yes!**

$$H^3_P(T^2) = F^1H^3_P(T^2) \supset F^2H^3_P(T^2) \supset F^3H^3_P(T^2)$$

- Our computations of $K_P^{\tau+n}(T^2)$ cover all the twists classified by $F^2H^3_P(T^2)$ but not all the twists on T^2.

Proposition(answer to the 1st question)

label	P	ori	$F^2H^3_P(T^2)$	basis
p1	1	$+$	0	
p2	$\mathbb{Z}_2$	$+$	0	
p3	$\mathbb{Z}_3$	$+$	0	
p4	$\mathbb{Z}_4$	$+$	0	
p6	$\mathbb{Z}_6$	$+$	0	
pm/pg	D_1	$-$	$\mathbb{Z}_2$	τ_{pg}
cm	D_1	$-$	0	
pmm/pmg/pgg	D_2	$-$	$\mathbb{Z}_2^{\oplus 3}$	$\tau_{\mathrm{pmg}}, \tau_{\mathrm{pgg}}, c$
cmm	D_2	$-$	$\mathbb{Z}_2$	c
p3m1	D_3	$-$	0	
p31m	D_3	$-$	0	
p4m/p4g	D_4	$-$	$\mathbb{Z}_2^{\oplus 2}$	τ_{p4g}, c
p6m	D_6	$-$	$\mathbb{Z}_2$	c

Proposition(answer to the 2nd question)

label	P	ori	$H^3_P(T^2)$	H^3/F^2H^3
p1	1	$+$	0	0
p2	$\mathbb{Z}_2$	$+$	0	0
p3	$\mathbb{Z}_3$	$+$	0	0
p4	$\mathbb{Z}_4$	$+$	0	0
p6	$\mathbb{Z}_6$	$+$	0	0
pm/pg	D_1	$-$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2$
cm	D_1	$-$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
pmm/pmg/pgg	D_2	$-$	$\mathbb{Z}_2^{\oplus 4}$	$\mathbb{Z}_2$
cmm	D_2	$-$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2$
p3m1	D_3	$-$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
p31m	D_3	$-$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
p4m/p4g	D_4	$-$	$\mathbb{Z}_2^{\oplus 3}$	$\mathbb{Z}_2$
p6m	D_6	$-$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2$

Equivariant cohomology degree up to 3

label	P	$H^1_P(T^2)$	$H^2_P(T^2)$	$H^3_P(T^2)$
p1	1	$\mathbb{Z}^{\oplus 2}$	$\mathbb{Z}$	0
p2	$\mathbb{Z}_2$	0	$\mathbb{Z} \oplus \mathbb{Z}_2^{\oplus 3}$	0
p3	$\mathbb{Z}_3$	0	$\mathbb{Z} \oplus \mathbb{Z}_3^{\oplus 2}$	0
p4	$\mathbb{Z}_4$	0	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$	0
p6	$\mathbb{Z}_6$	0	$\mathbb{Z} \oplus \mathbb{Z}_6$	0
pm/pg	D_1	$\mathbb{Z}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$
cm	D_1	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
pmm/pmg/pgg	D_2	0	$\mathbb{Z}_2^{\oplus 4}$	$\mathbb{Z}_2^{\oplus 4}$
cmm	D_2	0	$\mathbb{Z}_2^{\oplus 3}$	$\mathbb{Z}_2^{\oplus 2}$
p3m1	D_3	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
p31m	D_3	0	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$
p4m/p4g	D_4	0	$\mathbb{Z}_2^{\oplus 3}$	$\mathbb{Z}_2^{\oplus 3}$
p6m	D_6	0	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$