Quantising proper actions on Spin^c-manifolds

Peter Hochs

University of Adelaide

Differential geometry seminar Adelaide, 31 July 2015

Joint work with Mathai Varghese

- (symplectic case) "Geometric quantization and families of inner products", Adv. Math. (to appear), ArXiv:1309.6760
- (Spin^c-case) "Quantising proper actions on Spin^c-manifolds", ArXiv:1408.0085

Background: the compact, symplectic case

Spin^c-quantisation

Noncompact groups and manifolds

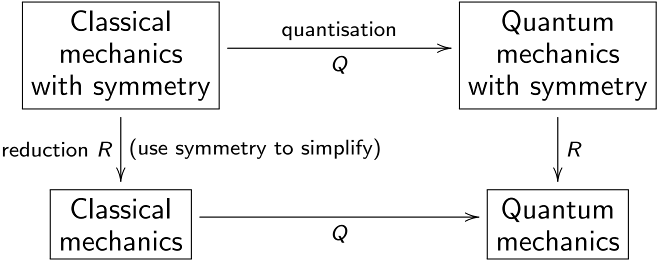
An analytic approach in the compact case

An analytic approach in the noncompact case

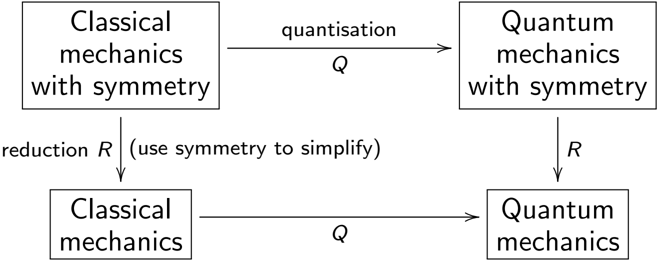
Twisted Spin^c-Dirac operators

1. Background

Quantisation and reduction



Quantisation and reduction



Quantisation commutes with reduction:

$$Q \circ R = R \circ Q, \quad \text{or} \quad `[Q, R] = 0'.$$

Setup

Let

- (M, ω) be a symplectic manifold;
- G be a Lie group, acting on M, preserving ω;
- J be a G-invariant almost complex structure on M, such that $\omega(-, J\,-)$ is a Riemannian metric;
- $L \to M$ be a Hermitian G-line bundle with a Hermitian connection ∇^L such that

$$(\nabla^L)^2 = 2\pi i \omega.$$

The Spin^c-Dirac operator

In this setting, one has the Spin^c-**Dirac operator**

$$\boxed{D := \sum_{j=1}^{\dim(M)} c(e_j)\nabla_{e_j}} \quad \text{on } \Omega^{0,*}(M; L)$$

where

- $\{e_1, \ldots, e_{\dim(M)}\}$ is a local orthonormal frame for TM;
- ∇ is a connection on $\bigwedge^{0,*} T^*M \otimes L$ induced by the Levi–Civita connection on TM and the connection ∇^L on L;
- $c(v) = \sqrt{2}\big(-i_{v^{0,1}} + (\overline{v^{1,0}})^* \wedge -\big)$, for $v \in T_m M$, is the Clifford action.

The Spin^c-Dirac operator

In this setting, one has the Spin^c-**Dirac operator**

$$\boxed{D := \sum_{j=1}^{\dim(M)} c(e_j)\nabla_{e_j}} \quad \text{on } \Omega^{0,*}(M;L)$$

where

- $\{e_1, \ldots, e_{\dim(M)}\}$ is a local orthonormal frame for TM;
- ∇ is a connection on $\bigwedge^{0,*} T^*M \otimes L$ induced by the Levi–Civita connection on TM and the connection ∇^L on L;
- $c(v) = \sqrt{2}\left(-i_{v^{0,1}} + (\overline{v^{1,0}})^* \wedge -\right)$, for $v \in T_mM$, is the Clifford action.

The operator D is elliptic, so $\ker D$ is finite-dimensional if M is **compact**.

Geometric quantisation

Suppose M and G are **compact**.

Definition (Bott)

The **geometric quantisation** of the action by G on (M, ω) is the equivariant index of D:

$$Q_G(M, \omega) = G\text{-index}(D) = [\ker D^+] - [\ker D^-] \quad \in R(G).$$

Here $D^\pm$ are the restrictions of D to the even and odd parts of $\bigwedge^{0,*} T^*M \otimes L$, and

$$R(G) := \left\{[V] - [W];\ V, W \text{ finite-dim. representation of } G\right\}$$

is the representation ring of G.

Quantum reduction

On the quantum side, reduction means taking the G-invariant part of a representation:

$$Q_G(M, \omega)^G = \dim(\ker D^+)^G - \dim(\ker D^-)^G \quad \in \mathbb{Z}.$$

(Here we identify equivalence classes of finite-dimensional vector spaces with their dimensions.)

Quantum reduction

On the quantum side, reduction means taking the G-invariant part of a representation:

$$Q_G(M,\omega)^G = \dim(\ker D^+)^G - \dim(\ker D^-)^G \quad \in \mathbb{Z}.$$

(Here we identify equivalence classes of finite-dimensional vector spaces with their dimensions.)

One can also take multiplicities of other irreducible representations than the trivial one.

Classical reduction

Suppose the action by G on (M, ω) is **Hamiltonian**, i.e. there is a **moment(um) map**

$$\mu : M \to \mathfrak{g}^*,$$

which is equivariant w.r.t. the coadjoint action by G on $\mathfrak{g}^*$, such that for all $X \in \mathfrak{g}$,

$$\boxed{2\pi i \langle \mu, X \rangle = \nabla^L_{X^M} - \mathcal{L}^L_X} \quad \in \mathrm{End}(L) = C^\infty(M, \mathbb{C}),$$

with X^M the vector field induced by X, and $\mathcal{L}^L$ the Lie derivative of sections of L.

Classical reduction

Suppose the action by G on (M, ω) is **Hamiltonian**, i.e. there is a **moment(um) map**

$$\mu : M \to \mathfrak{g}^*,$$

which is equivariant w.r.t. the coadjoint action by G on $\mathfrak{g}^*$, such that for all $X \in \mathfrak{g}$,

$$\boxed{2\pi i \langle \mu, X \rangle = \nabla^L_{X^M} - \mathcal{L}^L_X} \quad \in \mathrm{End}(L) = C^\infty(M, \mathbb{C}),$$

with X^M the vector field induced by X, and $\mathcal{L}^L$ the Lie derivative of sections of L.

Definition

Let $\xi \in \mathfrak{g}^*$. The **symplectic reduction** of the action is the space

$$\boxed{M_\xi := \mu^{-1}(\xi)/G_\xi}$$

Theorem (Marsden–Weinstein, 1974)

If ξ is a regular value of μ, and G_ξ acts freely (properly) on $\mu^{-1}(\xi)$, then M_ξ is a symplectic manifold (orbifold).

Quantisation commutes with reduction

For compact M and G, and M_0 smooth, the diagram on the first slide becomes:

$$\begin{array}{ccc}
G \circlearrowright (M,\omega) & \xmapsto{\quad Q \quad} & [\ker D^+] - [\ker D^-] \\
\Big\downarrow R & & \Big\downarrow R \\
 & & \dim(\ker D)^G - \dim(\ker D^-)^G \\
 & & \Big\| \, ? \\
(M_0,\omega_0) & \xmapsto{\quad Q \quad} & \dim(\ker D^+_{M_0}) - \dim(\ker(D_{M_0})^-)
\end{array}$$

Quantisation commutes with reduction

For compact M and G, and M_0 smooth, the diagram on the first slide becomes:

$$
\begin{array}{ccc}
G \circlearrowright (M,\omega) & \xmapsto{\quad Q \quad} & [\ker D^+] - [\ker D^-] \\
 & & \big\downarrow R \\
R \big\downarrow & & \dim(\ker D)^G - \dim(\ker D^-)^G \\
 & & \Big\| \; ? \\
(M_0,\omega_0) & \xmapsto{\; Q \;} & \dim(\ker D^+_{M_0}) - \dim(\ker(D_{M_0})^-)
\end{array}
$$

Proved by Meinrenken in 1995, and later by Paradan and Tian–Zhang, after a conjecture by Guillemin–Sternberg in 1982. Contributions by many others.

Quantisation commutes with reduction

For compact M and G, and M_0 smooth, the diagram on the first slide becomes:

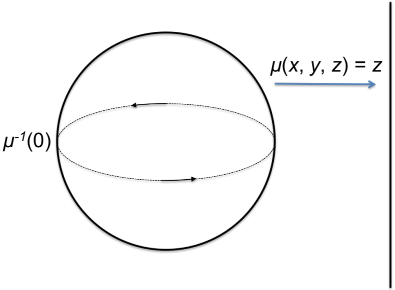
$$
\begin{array}{ccc}
G \circlearrowright (M,\omega) & \xmapsto{\quad Q \quad} & [\ker D^+] - [\ker D^-] \\
 & & \big\downarrow R \\
\Bigg\downarrow R & & \dim(\ker D)^G - \dim(\ker D^-)^G \\
 & & \Big\| \, ? \\
(M_0, \omega_0) & \xmapsto{\; Q \;} & \dim(\ker D^+_{M_0}) - \dim(\ker (D_{M_0})^-)
\end{array}
$$

Proved by Meinrenken in 1995, and later by Paradan and Tian–Zhang, after a conjecture by Guillemin–Sternberg in 1982. Contributions by many others.

Important step in proofs: **localisation** to $\mu^{-1}(0)$.

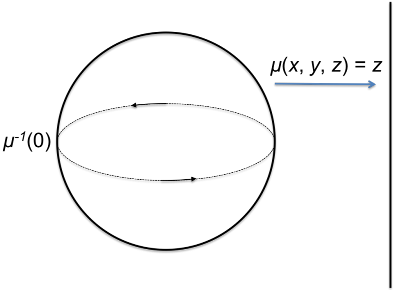
Compact example: spin

$G = \mathrm{SO}(2)$ acting on $M = S^2$ by rotations around the z-axis



Compact example: spin

$G = SO(2)$ acting on $M = S^2$ by rotations around the z-axis

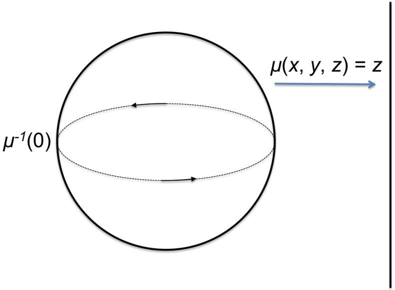


Now G and M are **compact**, and M_0 is a point. Hence

$$Q_G(M, \omega)^G = Q(M_0, \omega_0) = 1.$$

Compact example: spin

$G = \mathrm{SO}(2)$ acting on $M = S^2$ by rotations around the z-axis



Now G and M are **compact**, and M_0 is a point. Hence

$$Q_G(M, \omega)^G = Q(M_0, \omega_0) = 1.$$

Physics: direction of angular momentum in rotationally invariant potential

Maths: restricting irreducible representations of SO(3) to a maximal torus

2. Spin^c-quantisation

From symplectic to Spin^c

All symplectic manifolds are Spin^c, and only the Spin^c-structure is necessary to define a Dirac operator and quantisation.

So natural **question:** can one state and prove $[Q, R] = 0$ for Spin^c-manifolds in general? (Asked by Cannas da Silva, Karshon and Tolman in 2000, and answered for circle actions.)

This was generalised to actions by arbitrary compact, connected Lie groups by Paradan and Vergne in 2014.

Spin^c-Dirac operators

Let M be a Spin^c-manifold. Let $L \to M$ be the associated determinant linde bundle, and $\mathcal{S} \to M$ the spinor bundle. (If M is almost complex, then $\mathcal{S} = \bigwedge^{0,*} T^*M$.)

Spin^c-Dirac operators

Let M be a Spin^c-manifold. Let $L \to M$ be the associated determinant linde bundle, and $\mathcal{S} \to M$ the spinor bundle. (If M is almost complex, then $\mathcal{S} = \bigwedge^{0,*} T^*M$.)

The Levi–Civita connection on TM and a connection on L induce a connection on $\mathcal{S}$ via local decompositions

$$\mathcal{S}|_U \cong \mathcal{S}_0^U \otimes L|_U^{1/2},$$

where $\mathcal{S}_0^U \to U$ is the spinor bundle associated to a local Spin-structure.

This induces a Spin^c-Dirac operator D on $\Gamma^\infty(\mathcal{S})$.

Spin^c-reduction

Let G be a Lie group acting properly on M, and suppose the action lifts to the Spin^c-structure.

Spin^c-reduction

Let G be a Lie group acting properly on M, and suppose the action lifts to the Spin^c-structure.

As in the symplectic case, one can define a Spin^c-**momentum map** $\mu : M \to \mathfrak{g}^*$ by

$$2\pi i \langle \mu, X \rangle = \nabla^L_{X^M} - \mathcal{L}^L_X,$$

for $X \in \mathfrak{g}$.

Spin^c-reduction

Let G be a Lie group acting properly on M, and suppose the action lifts to the Spin^c-structure.

As in the symplectic case, one can define a Spin^c-**momentum map** $\mu : M \to \mathfrak{g}^*$ by

$$2\pi i \langle \mu, X \rangle = \nabla^L_{X^M} - \mathcal{L}^L_X,$$

for $X \in \mathfrak{g}$.

Definition

The **reduced space** at $\xi \in \mathfrak{g}^*$ is

$$M_\xi := \mu^{-1}(\xi)/G_\xi.$$

If G is a torus and ξ is a regular value of μ, then M_ξ is a Spin^c-orbifold. More generally, there is a (nontrivial) way to define $Q(M_\xi) \in \mathbb{Z}$.

Compact groups and manifolds

Suppose that M and G are compact and connected, and that M is even-dimensional. Then

$$Q_G(M) := G\text{-index}(D) = \sum_{\pi \in \hat{G}} m_\pi \pi,$$

for certain $m_\pi \in \mathbb{Z}$. Paradan and Vergne computed m_π in terms of quantisations of reduced spaces.

Compact groups and manifolds

Suppose that M and G are compact and connected, and that M is even-dimensional. Then

$$Q_G(M) := G\text{-index}(D) = \sum_{\pi \in \hat{G}} m_\pi \pi,$$

for certain $m_\pi \in \mathbb{Z}$. Paradan and Vergne computed m_π in terms of quantisations of reduced spaces.

Theorem (Paradan–Vergne, 2014)

Let λ be the highest weight of π, and ρ half the sum of the positive roots.

- *If the minimal stabiliser of the action is Abelian, then*

$$m_\pi = Q(M_{\lambda+\rho}).$$

- *In general, m_π is expressed as a finite* **sum** *of quantisations of reduced spaces.*

3. Noncompact groups and manifolds

The noncompact setting

Natural question: can this be generalised to **noncompact** G and M?

- Would give insight in representation theory of noncompact groups.
- Many phase spaces in classical mechanics (symplectic manifolds) are noncompact, e.g. cotangent bundles.
- In general, equivariant index theory of Spin^c-Dirac operators is a relevant subject.

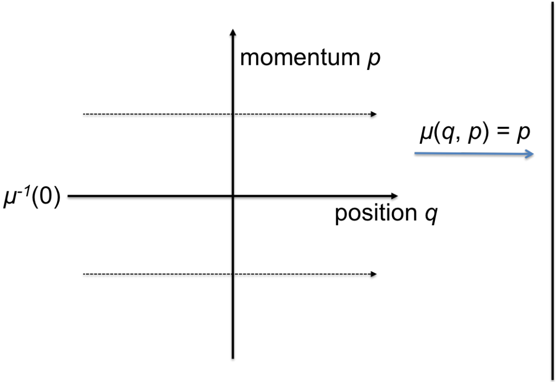
Existing results

There are $[Q, R] = 0$ results when either G or M/G may be noncompact.

- For G **compact** and μ proper, there is a result in the symplectic case by Ma–Zhang, with another proof given by Paradan. This was generalised to Spin^c-manifolds by H.–Song.
- For M/G **compact**, Landsman formulated a $[Q, R] = 0$ conjecture, in the symplectic case. Results in this context were obtained by Landsman, H., and Mathai–Zhang.

Noncompact example: free particle on a line

$G = \mathbb{R}$ acting on $M = \mathbb{R}^2$ by addition on the first coordinate



Now G, M and M/G are **noncompact**, so outside the scope of the Ma–Zhang/Paradan and Landsman approaches.

Goals and method

Goals: generalise $[Q, R] = 0$ to cases where

1. M, G and M/G may be noncompact;
2. M is only Spin^c;
3. the Spin^c-Dirac operator is twisted by an arbitrary vector bundle over M.

Method: generalise the analytic approach developed by Tian–Zhang.

4. An analytic approach in the compact case

Localising and decomposing

Consider the symplectic setting, and suppose M and G are compact.

Idea:

1. consider a deformed version D_t of the Dirac operator D, with a real deformation parameter t;
2. localise the G-invariant part of the kernel of D_t to a neighbourhood U of $\mu^{-1}(0)$ in a suitable sense, for t large enough;
3. on U, decompose D_t into a part on $\mu^{-1}(0)$ and a part normal to $\mu^{-1}(0)$.

Localising and decomposing

Consider the symplectic setting, and suppose M and G are compact.

Idea:

1. consider a deformed version D_t of the Dirac operator D, with a real deformation parameter t;
2. localise the G-invariant part of the kernel of D_t to a neighbourhood U of $\mu^{-1}(0)$ in a suitable sense, for t large enough;
3. on U, decompose D_t into a part on $\mu^{-1}(0)$ and a part normal to $\mu^{-1}(0)$.

In this talk we focus on the **localisation** of the kernel of D_t, since noncompactness plays the biggest role in that step.

Deforming the Dirac operator

Tian and Zhang used an $\mathrm{Ad}^*(G)$-invariant inner product on $\mathfrak{g}^*$, which exists for compact groups G. Then one has

$$\mu^* : M \to \mathfrak{g}$$

dual to μ.

Deforming the Dirac operator

Tian and Zhang used an $\mathrm{Ad}^*(G)$-invariant inner product on $\mathfrak{g}^*$, which exists for compact groups G. Then one has

$$\mu^* : M \to \mathfrak{g}$$

dual to μ.

Consider the vector field v given by

$$v_m := 2(\mu^*(m))^M_m,$$

and the **deformed** Spin^c**-Dirac operator**

$$\boxed{D_t := D + t\frac{\sqrt{-1}}{2}c(v),}$$

for $t \in \mathbb{R}$.

A Bochner-type formula

Theorem (Tian–Zhang)

On $\Omega^{0,*}(M; L)^G$, one has

$$\boxed{D_t^2 = D^2 + tA + 4\pi t\|\mu\|^2 + \frac{t^2}{4}\|v\|^2,}$$

where A is a vector bundle endomorphism.

A Bochner-type formula

Theorem (Tian–Zhang)

On $\Omega^{0,*}(M;L)^G$, one has

$$\boxed{D_t^2 = D^2 + tA + 4\pi t\|\mu\|^2 + \frac{t^2}{4}\|v\|^2,}$$

where A is a vector bundle endomorphism.

Together with an explicit expression for A, and harmonic oscillator-type estimates for A, this allowed Tian and Zhang to localise $(\ker D_t)^G$ to $\mu^{-1}(0)$ for large t, and prove that $[Q,R]=0$.

5. An analytic approach in the noncompact case

The noncompact case

Idea: generalise Tian–Zhang's localisation arguments both to define quantisation and to prove $[Q, R] = 0$.

The noncompact case

Idea: generalise Tian–Zhang's localisation arguments both to define quantisation and to prove $[Q, R] = 0$.

Issues:

- If G is noncompact, there is no $\mathrm{Ad}^*(G)$-invariant inner product on $\mathfrak{g}^*$ in general, so the deformed Dirac operator may not be G-equivariant.
- If M is noncompact, the operator A in

$$D_t^2 = D^2 + tA + 4\pi t\|\mu\|^2 + \frac{t^2}{4}\|v\|^2,$$

  may be unbounded. (And $\|\mu\|^2$ and v may go to zero at infinity.)
- If M is only Spin^c, the expression for the operator A becomes less explicit.

The noncompact case

Idea: generalise Tian–Zhang's localisation arguments both to define quantisation and to prove $[Q, R] = 0$.

Issues:

- If G is noncompact, there is no $\mathrm{Ad}^*(G)$-invariant inner product on $\mathfrak{g}^*$ in general, so the deformed Dirac operator may not be G-equivariant.
- If M is noncompact, the operator A in
$$D_t^2 = D^2 + tA + 4\pi t\|\mu\|^2 + \frac{t^2}{4}\|v\|^2,$$
may be unbounded. (And $\|\mu\|^2$ and v may go to zero at infinity.)
- If M is only Spin^c, the expression for the operator A becomes less explicit.

Solution: use **families of inner products** on $\mathfrak{g}^*$, parametrised by M.

Families of inner products

Let $\{(-,-)_m\}_{m\in M}$ be a smooth family of inner products on $\mathfrak{g}^*$, with the invariance property that for all $m \in M$, $g \in G$ and $\xi, \xi' \in \mathfrak{g}^*$,

$$\left(\mathrm{Ad}^*(g)\xi, \mathrm{Ad}^*(g)\xi'\right)_{g\cdot m} = (\xi, \xi')_m.$$

Families of inner products

Let $\{(-,-)_m\}_{m\in M}$ be a smooth family of inner products on $\mathfrak{g}^*$, with the invariance property that for all $m \in M$, $g \in G$ and $\xi, \xi' \in \mathfrak{g}^*$,

$$\left(\mathrm{Ad}^*(g)\xi, \mathrm{Ad}^*(g)\xi'\right)_{g\cdot m} = (\xi, \xi')_m.$$

Then we define

- the map $\mu^* : M \to \mathfrak{g}$ by

$$\langle \xi, \mu^*(m)\rangle = (\xi, \mu(m))_m$$

  for all $\xi \in \mathfrak{g}^*$ and $m \in M$;
- the vector field v as before $v_m := 2(\mu^*(m))^M_m$;
- the deformed Dirac operator (which is **equivariant**)

$$D_v = D + \frac{\sqrt{-1}}{2}c(v).$$

  (Now the parameter t can be absorbed into the family of inner products.)

Main assumption

We defined:

$$v_m := 2(\mu^*(m))^M_m.$$

The main assumption is that $\mathrm{Zeros}(v)/G$ is **compact**. Since $\mu^{-1}(0) \subset \mathrm{Zeros}(v)$, this implies that M_0 is compact.

Main assumption

We defined:

$$v_m := 2(\mu^*(m))^M_m.$$

The main assumption is that $\mathrm{Zeros}(v)/G$ is **compact**. Since $\mu^{-1}(0) \subset \mathrm{Zeros}(v)$, this implies that M_0 is compact.

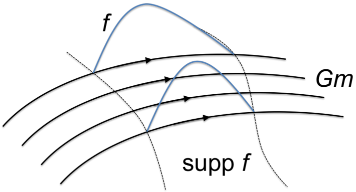
Other assumptions: G is **unimodular**, and acts **properly** on M.

Transversally L^2-sections

We will use **transversally** L^2-sections to define an index of D_v.

- A **cutoff function** is a function $f \in C^\infty(M)$ such that for all $m \in M$, the intersection $G \cdot m \cap \mathrm{supp}(f)$ is compact, and for a Haar measure dg on G,

$$\boxed{\int_G f(g \cdot m)^2 \, dg = 1.}$$

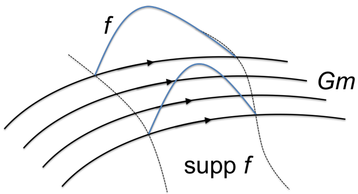


Transversally L^2-sections

We will use **transversally** L^2-sections to define an index of D_v.

- A **cutoff function** is a function $f \in C^\infty(M)$ such that for all $m \in M$, the intersection $G \cdot m \cap \operatorname{supp}(f)$ is compact, and for a Haar measure dg on G,

$$\boxed{\int_G f(g \cdot m)^2 \, dg = 1.}$$



- For any vector bundle $E \to M$ equipped with a metric, the space of G-**invariant, transversally** L^2**-sections** of E is

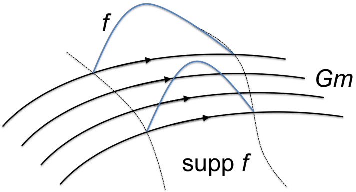
$$L^2_T(E)^G := \{s \in \Gamma(E)^G; fs \in L^2(E) \text{ for a cutoff function } f\}/ =_{\text{a.e.}} .$$

Transversally L^2-sections

We will use **transversally** L^2-sections to define an index of D_v.

- A **cutoff function** is a function $f \in C^\infty(M)$ such that for all $m \in M$, the intersection $G \cdot m \cap \mathrm{supp}(f)$ is compact, and for a Haar measure dg on G,

$$\boxed{\int_G f(g \cdot m)^2 \, dg = 1.}$$



- For any vector bundle $E \to M$ equipped with a metric, the space of G-**invariant, transversally** L^2**-sections** of E is

$$L^2_T(E)^G := \{s \in \Gamma(E)^G; fs \in L^2(E) \text{ for a cutoff function } f\}/ =_{\text{a.e.}} .$$

- For any (differential) operator D on $\Gamma^\infty(E)$, we have the G-**invariant, transversally** L^2**-kernel** of D:

$$(\ker_{L^2_T} D)^G := \{s \in \Gamma^\infty(E) \cap L^2_T(E)^G; Ds = 0\}.$$

Special cases

- If G **is compact**, $f \equiv 1$ is a cutoff function, so
$$L^2_T(E)^G = L^2(E)^G.$$
- If M/G **is compact**, cutoff functions have compact supports, so
$$\Gamma^\infty(E) \cap L^2_T(E) = \Gamma^\infty(E).$$

Special cases

- If G **is compact**, $f \equiv 1$ is a cutoff function, so
$$L^2_T(E)^G = L^2(E)^G.$$
- If M/G **is compact**, cutoff functions have compact supports, so
$$\Gamma^\infty(E) \cap L^2_T(E) = \Gamma^\infty(E).$$

In general, $L^2(E)^G$ is independent of the cutoff function used, by unimodularity of G.

Invariant quantisation

We had the deformed Dirac operator

$$D_v = D + \frac{\sqrt{-1}}{2} c(v).$$

Invariant quantisation

We had the deformed Dirac operator

$$D_v = D + \frac{\sqrt{-1}}{2} c(v).$$

Theorem (Mathai–H. in symplectic case; Braverman for general Dirac-type operators)

If $\mathrm{Zeros}(v)/G$ *is compact, then the metric on* $M \times \mathfrak{g}^*$ *can be rescaled by a positive function so that*

$$\dim\left(\ker_{L^2_T} D_v\right)^G < \infty.$$

Invariant quantisation

We had the deformed Dirac operator

$$D_v = D + \frac{\sqrt{-1}}{2} c(v).$$

Theorem (Mathai–H. in symplectic case; Braverman for general Dirac-type operators)

If $\mathrm{Zeros}(v)/G$ *is compact, then the metric on* $M \times \mathfrak{g}^*$ *can be rescaled by a positive function so that*

$$\dim\left(\ker_{L^2_T} D_v\right)^G < \infty.$$

Definition

The G**-invariant, transversally** L^2**-index** of D_v is

$$\mathrm{index}^G_{L^2_T} D_v := \dim\left(\ker_{L^2_T} D_v^+\right)^G - \dim\left(\ker_{L^2_T} D_v^-\right)^G.$$

Quantisation commutes with reduction

Theorem (Mathai–H., 2014)

Suppose 0 *is a regular value of* μ, *and* G *acts freely on* $\mu^{-1}(0)$. *Then there is a class of* Spin^c*-structures on* M, *such that*

$$\mathrm{index}_{L^2_T}^G D_v = \mathrm{index}\, D_{M_0}$$

Quantisation commutes with reduction

Theorem (Mathai–H., 2014)

Suppose 0 *is a regular value of* μ, *and* G *acts freely on* $\mu^{-1}(0)$. *Then there is a class of* Spin^c*-structures on* M, *such that*

$$\mathrm{index}^G_{L^2_T} D_v = \mathrm{index}\, D_{M_0}$$

The class of Spin^c-structures in the theorem corresponds to using high enough tensor powers of the determinant line bundle.

Quantisation commutes with reduction

Theorem (Mathai–H., 2014)

Suppose 0 *is a regular value of* μ, *and* G *acts freely on* $\mu^{-1}(0)$. *Then there is a class of* Spin^c*-structures on* M, *such that*

$$\mathrm{index}^G_{L^2_T} D_v = \mathrm{index}\, D_{M_0}$$

The class of Spin^c-structures in the theorem corresponds to using high enough tensor powers of the determinant line bundle.

In the symplectic analogue of this result, one does not need high tensor powers of the line bundle if

- G is compact, or
- the action is locally free.

A Bochner formula for families of inner products

As in the compact case, the proofs of the results start with an explicit expression for D_v^2.

A Bochner formula for families of inner products

As in the compact case, the proofs of the results start with an explicit expression for D_v^2.

Theorem

On $\Gamma^\infty(\mathcal{S})^G$, one has

$$D_v^2 = D^2 + A + 2\pi\|\mu\|^2 + \frac{1}{4}\|v\|^2,$$

with A a vector bundle endomorphism.

A Bochner formula for families of inner products

As in the compact case, the proofs of the results start with an explicit expression for D_v^2.

Theorem

On $\Gamma^\infty(\mathcal{S})^G$, one has

$$D_v^2 = D^2 + A + 2\pi\|\mu\|^2 + \frac{1}{4}\|v\|^2,$$

with A a vector bundle endomorphism.

The expression for A is different from the compact case, because of

- extra terms due to the use of a family of inner products on $\mathfrak{g}^*$;
- the fact that M is only assumed to be Spin^c.

In addition, one has no control over the behaviour of A, $\|\mu\|^2$ and $\|v\|$ 'at infinity'.

Choosing the family of inner products on $\mathfrak{g}^*$

Solution to issues arising in the noncompact/Spin^c-case: a suitable choice of the family of inner products on $\mathfrak{g}^*$.

Choosing the family of inner products on $\mathfrak{g}^*$

Solution to issues arising in the noncompact/Spin^c-case: a suitable choice of the family of inner products on $\mathfrak{g}^*$. Let

- V be a G-invariant, relatively cocompact neighbourhood of Zeros(v)
- η be any G-invariant smooth function on M

Proposition

The family of inner products on $\mathfrak{g}^*$ can be rescaled by a positive function in such a way that for all $m \in M \setminus V$,

$$\|\mu(m)\|^2 \geq 1;$$
$$\|v_m\| \geq \eta(m),$$

and there is a $C > 0$, such that for all $m \in M$,

$$A_m \geq -C(\|v_m\|^2 + 1).$$

Choosing the family of inner products on $\mathfrak{g}^*$

Solution to issues arising in the noncompact/Spin^c-case: a suitable choice of the family of inner products on $\mathfrak{g}^*$. Let

- V be a G-invariant, relatively cocompact neighbourhood of $\mathrm{Zeros}(v)$
- η be any G-invariant smooth function on M

Proposition

The family of inner products on $\mathfrak{g}^*$ can be rescaled by a positive function in such a way that for all $m \in M \setminus V$,

$$\|\mu(m)\|^2 \geq 1;$$
$$\|v_m\| \geq \eta(m),$$

and there is a $C > 0$, such that for all $m \in M$,

$$A_m \geq -C(\|v_m\|^2 + 1).$$

This turns out to be enough to localise $(\ker_{L^2_T} D_t)^G$, and get $[Q, R] = 0$.

5. Twisted Spin^c-Dirac operators

Twisting Dirac operators by vector bundles

As before, let M be a Riemannian manifolds, on which a Lie group G acts properly and isometrically. Suppose M has a G-equivariant Spin^c-structure, with spinor bundle $\mathcal{S} \to M$. Consider a G-invariant, Hermitian connection $\nabla^{\mathcal{S}}$ on $\mathcal{S}$.

Twisting Dirac operators by vector bundles

As before, let M be a Riemannian manifolds, on which a Lie group G acts properly and isometrically. Suppose M has a G-equivariant Spin^c-structure, with spinor bundle $\mathcal{S} \to M$. Consider a G-invariant, Hermitian connection $\nabla^{\mathcal{S}}$ on $\mathcal{S}$.

Now let $E \to M$ be a Hermitian, G-equivariant vector bundle, with a Hermitian, G-invariant connection ∇^E. Then we have the connection

$$\nabla^{\mathcal{S}\otimes E} := \nabla^{\mathcal{S}} \otimes 1_E + 1_{\mathcal{S}} \otimes \nabla^E$$

on $\mathcal{S} \otimes E$.

Definition

The Spin^c-**Dirac operator** $D^{\mathcal{S}\otimes E}$ **twised by** E **(via** ∇^E**)** is the operator on $\Gamma^\infty(\mathcal{S} \otimes E)$ locally given by

$$D^E = \sum_j (c(e_j) \otimes 1_E) \nabla^{\mathcal{S}\otimes E}_{e_j}.$$

Localising twisted Dirac operators

As before, we set

$$D_v^E := D^E + \frac{\sqrt{-1}}{2} c(v),$$

and suppose $\mathrm{Zeros}(v)/G$ is compact.

Theorem (H.–Mathai, 2015)

The metric on $M \times \mathfrak{g}^*$ can be rescaled such that, after replacing the determinant line bundle L by a high enough tensor power L^p,

$$\dim\left(\ker_{L^2_T} D_v\right)^G < \infty,$$

and

$$\mathrm{index}^G_{L^2_T} D_v = \mathrm{index}\, D^{E_0}_{M_0} = \int_{M_0} \mathrm{ch}(E_0) e^{\frac{p}{2} c_1(L_0)} \hat{A}(M_0).$$

Here $E_0 := (E|_{\mu^{-1}(0)})/G \to M_0$, and similarly for L_0.

Application 1: excision

Braverman defined an invariant, transversally L^2-index for general Dirac-type operators, deformed by a vector field v. As a consequence of a cobordism invariance property, this index is determined by data near $\mathrm{Zeros}(v)$.

Corollary

For a twisted Spin^c*-Dirac operator* D_v^E*, its index* $\mathrm{index}^G_{L^2_T} D_v$ *is determined by data in a neighbourhood of* $\mu^{-1}(0) \subset \mathrm{Zeros}(v)$*.*

Application 2: the signature operator

If M is Spin, then $D^{\mathcal{S}}$ equals the **signature operator** B on

$$\mathcal{S} \otimes \mathcal{S} \cong \bigwedge T^*M.$$

If M is only Spin^c, then $D^{\mathcal{S}}$ is the twisted signature operator B^L on

$$\bigwedge T^*M \otimes L.$$

Application 2: the signature operator

If M is Spin, then $D^{\mathcal{S}}$ equals the **signature operator** B on

$$\mathcal{S} \otimes \mathcal{S} \cong \bigwedge T^*M.$$

If M is only Spin^c, then $D^{\mathcal{S}}$ is the twisted signature operator B^L on

$$\bigwedge T^*M \otimes L.$$

Using this, one obtains

$$\mathrm{index}^G_{L^2_T} B^{L^p} = \int_{M_0} \mathrm{ch}(\mathcal{S}_0^N) e^{(p-\frac{1}{2})c_1(L_0)} L(M_0).$$

Here

- $\mathcal{S}^N \to \mu^{-1}(0)$ is the spinor bundle of the normal bundle $N \to \mu^{-1}(0)$ to TM_0;
- $L(M_0) = \mathrm{ch}(\mathcal{S}_{M_0})\hat{A}(M_0)$ is the L-class.

Thank you

Optional activities

- Colloquium "Dynamics on Networks: The role of local dynamics and global networks on hypersynchronous neural activity" by John Terry, 15:10, basement room B21
- Friday drinks at the staff club, meet at 17:00 in front of the lifts on the ground floor
- Dinner at the British on Finniss Street on Saturday (e.g. barbecued kangaroo)