Stretching String Topology

Tarje Bargheer

postdoc at University of Melbourne – supported by the Carlsberg-foundation

October 16, 2012

Mapping spaces

For M and N topological spaces, we consider the space

$$M^N := \{f\colon N \to M \mid f \text{ is continous}\}.$$

Mapping spaces

For M and N topological spaces, we consider the space

$$M^N := \{f\colon N \to M \mid f \text{ is continous}\}.$$

Theorem (Chas & Sullivan '99)

For M a compact, oriented d-manifold, letting $\mathbb{H}_*(-) := \mathrm{H}_{*+d}(-)$ denote the d-shifted homology,

$$\mathbb{H}_*\left(M^{S^1}\right)$$

carries the structure of a (2)-Batalin-Vilkovisky algebra.

Mapping spaces

For M and N topological spaces, we consider the space

$$M^N := \{f\colon N \to M \mid f \text{ is continous}\}.$$

Theorem (Chas & Sullivan '99)

For M a compact, oriented d-manifold, letting $\mathbb{H}_*(-) := \mathrm{H}_{*+d}(-)$ denote the d-shifted homology,

$$\mathbb{H}_*\left(M^{S^1}\right)$$

carries the structure of a (2)-Batalin-Vilkovisky algebra.

In particular, we have a multiplication

$$\mathbb{H}_*(M^{S^1}) \otimes \mathbb{H}_*(M^{S^1}) \to \mathbb{H}_*(M^{S^1})$$

Coloured Operads

A coloured topological operad O consist of

Coloured Operads

A coloured topological operad O consist of

- Spaces $O(-; k)$ of *k-ary morphisms* for all $k \in \mathbb{N}$.

Coloured Operads

A coloured topological operad O consist of

- Spaces $O(-; k)$ of *k-ary morphisms* for all $k \in \mathbb{N}$.
- A Space $\mathrm{Ob}(O)$ of colours.

Coloured Operads

A coloured topological operad $\mathcal{O}$ consist of

- Spaces $\mathcal{O}(-;k)$ of *k-ary morphisms* for all $k \in \mathbb{N}$.
- A Space $\mathrm{Ob}(\mathcal{O})$ of colours.
- For $k, m \in \mathbb{N}$ and $i \in \{1, \dots, k\}$, commutative diagrams

$$
\begin{array}{ccccc}
\mathcal{O}(-;k+m-1) & \xleftarrow{\circ_i} & \mathcal{O}(-;k) \times_{\mathrm{Ob}(\mathcal{O})} \mathcal{O}(-;m) & \longrightarrow & \mathcal{O}(-;m) \\
 & & \big\downarrow & & \big\downarrow \mathrm{ev}_{\mathrm{in}} \\
 & & \mathcal{O}(-;k) & \xrightarrow{\mathrm{ev}_i} & \mathrm{Ob}(\mathcal{O})
\end{array}
$$

along with certain coherence laws.

Coloured Operads

A coloured topological operad $\mathcal{O}$ consist of

- Spaces $\mathcal{O}(-;k)$ of *k-ary morphisms* for all $k \in \mathbb{N}$.
- A Space $\mathrm{Ob}(\mathcal{O})$ of colours.
- For $k, m \in \mathbb{N}$ and $i \in \{1, \dots, k\}$, commutative diagrams

$$\begin{array}{ccccc}
\mathcal{O}(-;k+m-1) & \xleftarrow{\circ_i} & \mathcal{O}(-;k) \times_{\mathrm{Ob}(\mathcal{O})} \mathcal{O}(-;m) & \longrightarrow & \mathcal{O}(-;m) \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_{\mathrm{in}}} \\
 & & \mathcal{O}(-;k) & \xrightarrow{\mathrm{ev}_i} & \mathrm{Ob}(\mathcal{O})
\end{array}$$

along with certain coherence laws.

Letting $k = m = 1$ gives categories internal to topological spaces.

Coloured Operads

A coloured topological operad $\mathcal{O}$ consist of

- Spaces $\mathcal{O}(-;k)$ of *k-ary morphisms* for all $k \in \mathbb{N}$.
- A Space $\mathrm{Ob}(\mathcal{O})$ of colours.
- For $k, m \in \mathbb{N}$ and $i \in \{1, \dots, k\}$, commutative diagrams

$$\begin{array}{ccccc} \mathcal{O}(-;k+m-1) & \xleftarrow{\circ_i} & \mathcal{O}(-;k) \times_{\mathrm{Ob}(\mathcal{O})} \mathcal{O}(-;m) & \to & \mathcal{O}(-;m) \\ & & \downarrow & & \downarrow \mathrm{ev_{in}} \\ & & \mathcal{O}(-;k) & \xrightarrow{\mathrm{ev}_i} & \mathrm{Ob}(\mathcal{O}) \end{array}$$

along with certain coherence laws.

Letting $k = m = 1$ gives categories internal to topological spaces. Taking $\mathrm{Ob}(\mathcal{O}) = *$ yields classical operads

The Little Disk Operad

An example operad is the Little n-Disk Operad.

The Little Disk Operad

An example operad is the Little *n*-Disk Operad. $\mathcal{D}isk_n$ has operadic data: $\mathrm{Ob}(\mathcal{D}isk_n) = *$

The Little Disk Operad

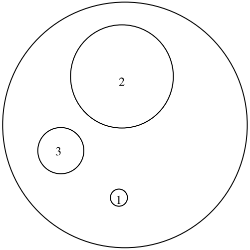
An example operad is the Little n-Disk Operad. $\mathcal{Disk}_n$ has operadic data: $\mathrm{Ob}(\mathcal{Disk}_n) = *$

$$\mathcal{Disk}_n(k) := \{\coprod_{i=1}^{k} f_i : \coprod_{i=1}^{k} D^n \hookrightarrow D^n \mid f_i \text{ is a dilation and translation}\}$$

The Little Disk Operad

An example operad is the Little n-Disk Operad. $\mathcal{Disk}_n$ has operadic data: $\mathrm{Ob}(\mathcal{Disk}_n) = *$

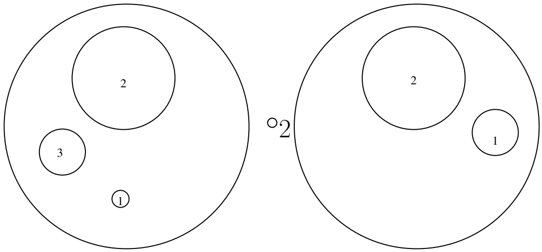
$$\mathcal{Disk}_n(k) := \{\coprod_{i=1}^{k} f_i : \coprod_{i=1}^{k} D^n \hookrightarrow D^n \mid f_i \text{ is a dilation and translation}\}$$



The Little Disk Operad

An example operad is the Little n-Disk Operad. $\mathcal{D}isk_n$ has operadic data: $\text{Ob}(\mathcal{D}isk_n) = *$

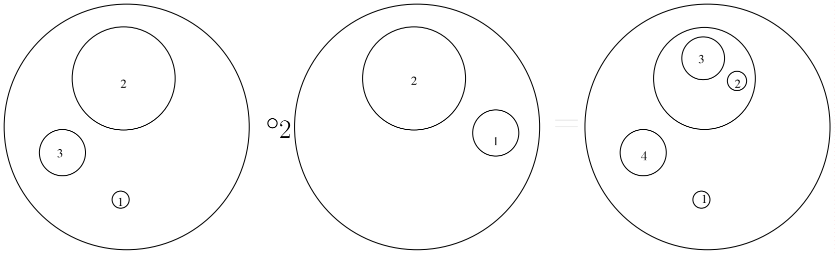
$$\mathcal{D}isk_n(k) := \{\coprod_{i=1}^{k} f_i : \coprod_{i=1}^{k} D^n \hookrightarrow D^n \mid f_i \text{ is a dilation and translation}\}$$



The Little Disk Operad

An example operad is the Little n-Disk Operad. $\mathcal{D}isk_n$ has operadic data: $\text{Ob}(\mathcal{D}isk_n) = *$

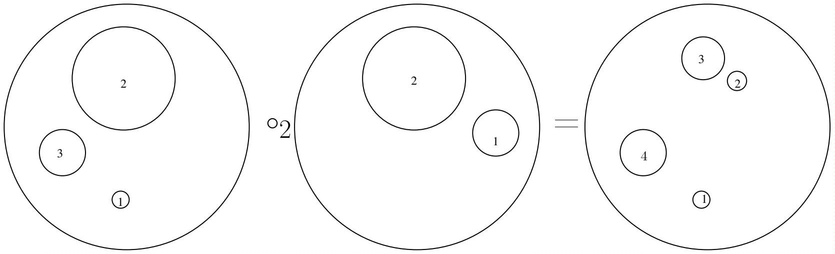
$$\mathcal{D}isk_n(k) := \{\coprod_{i=1}^{k} f_i : \coprod_{i=1}^{k} D^n \hookrightarrow D^n \mid f_i \text{ is a dilation and translation}\}$$



The Little Disk Operad

An example operad is the Little n-Disk Operad. $\mathcal{D}isk_n$ has operadic data: $\mathrm{Ob}(\mathcal{D}isk_n) = *$

$$\mathcal{D}isk_n(k) := \{\coprod_{i=1}^{k} f_i : \coprod_{i=1}^{k} D^n \hookrightarrow D^n \mid f_i \text{ is a dilation and translation}\}$$

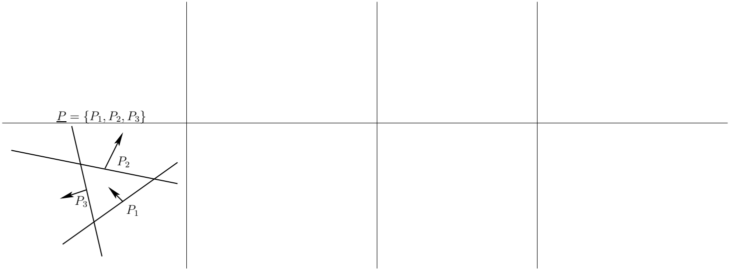


Cleaving Data:

Consider $(T, \underline{P})$ where T is a binary, rooted planar tree, and $\underline{P}$ a decoration at each internal vertex of T with an affine, oriented hyperplane of $\mathbb{R}^{n+1}$.

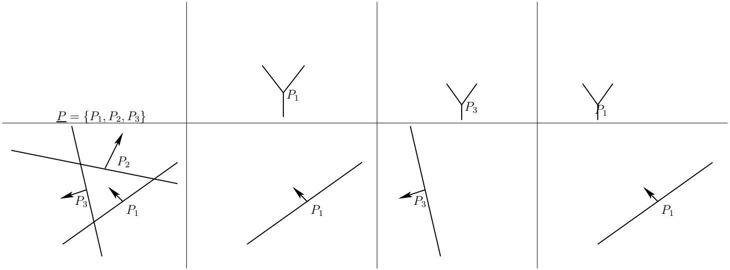
Cleaving Data:

Consider $(T, \underline{P})$ where T is a binary, rooted planar tree, and $\underline{P}$ a decoration at each internal vertex of T with an affine, oriented hyperplane of $\mathbb{R}^{n+1}$.



Cleaving Data:

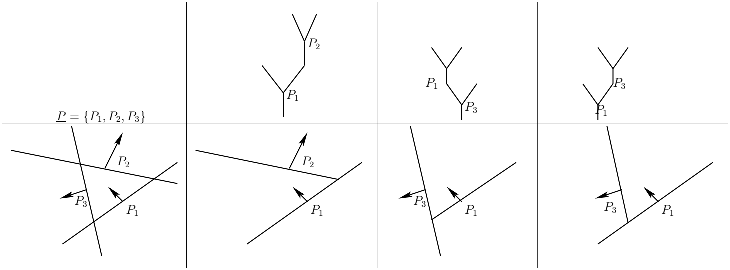
Consider $(T, \underline{P})$ where T is a binary, rooted planar tree, and $\underline{P}$ a decoration at each internal vertex of T with an affine, oriented hyperplane of $\mathbb{R}^{n+1}$.



Each hyperplane bisect $\mathbb{R}^{n+1}$ into a negative and a positive part

Cleaving Data:

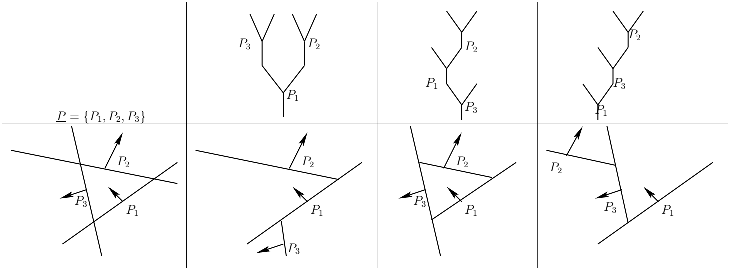
Consider $(T, \underline{P})$ where T is a binary, rooted planar tree, and $\underline{P}$ a decoration at each internal vertex of T with an affine, oriented hyperplane of $\mathbb{R}^{n+1}$.



For the respective direction of the tree, we use the decoration to cleave the corresponding bisection of $\mathbb{R}^{n+1}$ into new subspaces

Cleaving Data:

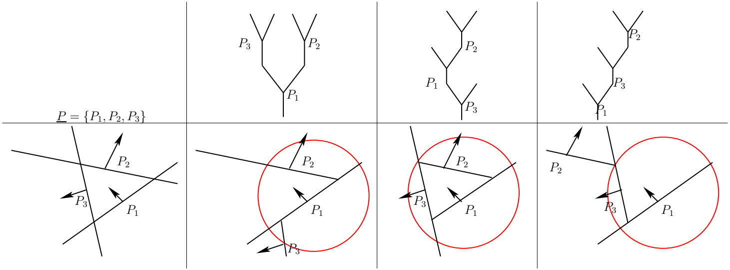
Consider $(T, \underline{P})$ where T is a binary, rooted planar tree, and $\underline{P}$ a decoration at each internal vertex of T with an affine, oriented hyperplane of $\mathbb{R}^{n+1}$.



The data $(T, \underline{P})$ cleave $\mathbb{R}^{n+1}$ if all hyperplanes intersect the subspace they cleave non-trivially

Cleaving Data:

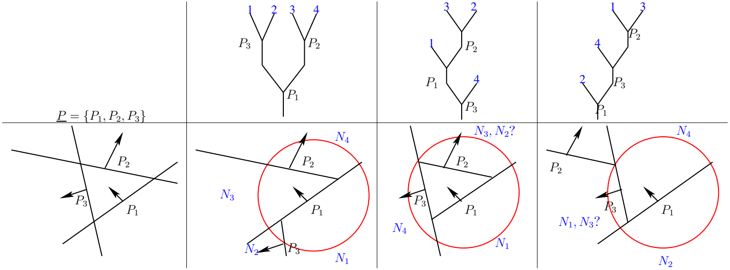
Consider $(T, \underline{P})$ where T is a binary, rooted planar tree, and $\underline{P}$ a decoration at each internal vertex of T with an affine, oriented hyperplane of $\mathbb{R}^{n+1}$.



The data $(T, \underline{P})$ cleave $N \subseteq \mathbb{R}^{n+1}$ a submanifold if the positive and negative part from all hyperplanes intersect the subspace of N they cleave non-trivially, and the hyperplanes are transverse to N

Cleaving Data:

Consider $(T, \underline{P})$ where T is a binary, rooted planar tree, and $\underline{P}$ a decoration at each internal vertex of T with an affine, oriented hyperplane of $\mathbb{R}^{n+1}$.



If $(T, \underline{P})$ cleave N, we assign the subsets of N that is the result of the cleaving procedure as timber to N. These are labelled according to the leafs of T

The Cleavage Operad

Any $N \subseteq \mathbb{R}^{n+1}$, defines a cleavage operad $\mathcal{Cleav}_N$:

The Cleavage Operad

Any $N \subseteq \mathbb{R}^{n+1}$, defines a cleavage operad $\mathcal{C}\ell eav_N$:

- $\mathrm{Ob}(\mathcal{C}\ell eav_N)$ is given by timber associated to any cleaving data $(T, \underline{P})$ of N

The Cleavage Operad

Any $N \subseteq \mathbb{R}^{n+1}$, defines a cleavage operad $\mathcal{Cleav}_N$:

- $\mathrm{Ob}(\mathcal{Cleav}_N)$ is given by timber associated to any cleaving data $(T, \underline{P})$ of N
- $\mathcal{Cleav}_N(U; k)$ will be a subquotient of the set

$$\Big\{(T, \underline{P}) \mid T \text{ is } k\text{-ary and } (T, \underline{P}) \text{ cleaves } U \in \mathrm{Ob}(\mathcal{Cleav}_N)\Big\}$$

The Cleavage Operad

Any $N \subseteq \mathbb{R}^{n+1}$, defines a cleavage operad $\mathcal{Cleav}_N$:

- $\mathrm{Ob}(\mathcal{Cleav}_N)$ is given by timber associated to any cleaving data $(T, \underline{P})$ of N
- $\mathcal{Cleav}_N(U; k)$ will be a subquotient of the set

$$\left\{ (T, \underline{P}) \mid T \text{ is } k\text{-ary and } (T, \underline{P}) \text{ cleaves } U \in \mathrm{Ob}(\mathcal{Cleav}_N) \right\}$$

We take the subset where each timber N_i of $(T, \underline{P})$ has $\complement N_i \simeq \coprod_{\text{finite}} *$.

The Cleavage Operad

Any $N \subseteq \mathbb{R}^{n+1}$, defines a cleavage operad $\mathcal{C}leav_N$:

- $\mathrm{Ob}(\mathcal{C}leav_N)$ is given by timber associated to any cleaving data $(T, \underline{P})$ of N
- $\mathcal{C}leav_N(U; k)$ will be a subquotient of the set

$$\Big\{(T, \underline{P}) \mid T \text{ is } k\text{-ary and } (T, \underline{P}) \text{ cleaves } U \in \mathrm{Ob}(\mathcal{C}leav_N)\Big\}$$

We take the subset where each timber N_i of $(T, \underline{P})$ has $\complement N_i \simeq \coprod_{\text{finite}} *$. Let $(T, \underline{P}) \sim (T', \underline{P}')$ if all their timber $N_1, \ldots, N_k$ and $N_1', \ldots, N_k'$ satisfiy $N_i = N_i'$ for all $i \in \{1, \ldots, k\}$

The Cleavage Operad

Any $N \subseteq \mathbb{R}^{n+1}$, defines a cleavage operad $\mathcal{C}\ell ea\mathcal{V}_N$:

- $\mathrm{Ob}(\mathcal{C}\ell ea\mathcal{V}_N)$ is given by timber associated to any cleaving data $(T, \underline{P})$ of N
- $\mathcal{C}\ell ea\mathcal{V}_N(U; k)$ will be a subquotient of the set

$$\left\{(T, \underline{P}) \mid T \text{ is } k\text{-ary and } (T, \underline{P}) \text{ cleaves } U \in \mathrm{Ob}(\mathcal{C}\ell ea\mathcal{V}_N)\right\}$$

We take the subset where each timber N_i of $(T, \underline{P})$ has $\complement N_i \simeq \coprod_{\text{finite}} *$. Let $(T, \underline{P}) \sim (T', \underline{P}')$ if all their timber $N_1, \ldots, N_k$ and $N'_1, \ldots, N'_k$ satisfiy $N_i = N'_i$ for all $i \in \{1, \ldots, k\}$

The above are given topology by seeing that $S^n \times \mathbb{R}$ parametrize affine, oriented hyperplanes.

The Cleavage Operad

Any $N \subseteq \mathbb{R}^{n+1}$, defines a cleavage operad $\mathcal{C}\mathit{leav}_N$:

- $\mathrm{Ob}(\mathcal{C}\mathit{leav}_N)$ is given by timber associated to any cleaving data $(T, \underline{P})$ of N
- $\mathcal{C}\mathit{leav}_N(U; k)$ will be a subquotient of the set

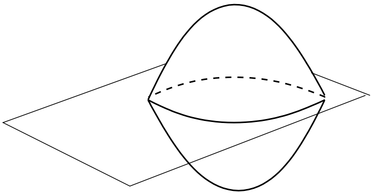
$$\Big\{(T, \underline{P}) \mid T \text{ is } k\text{-ary and } (T, \underline{P}) \text{ cleaves } U \in \mathrm{Ob}(\mathcal{C}\mathit{leav}_N)\Big\}$$

We take the subset where each timber N_i of $(T, \underline{P})$ has $\complement N_i \simeq \coprod_{\text{finite}} *$. Let $(T, \underline{P}) \sim (T', \underline{P}')$ if all their timber $N_1, \ldots, N_k$ and $N'_1, \ldots, N'_k$ satisfiy $N_i = N'_i$ for all $i \in \{1, \ldots, k\}$

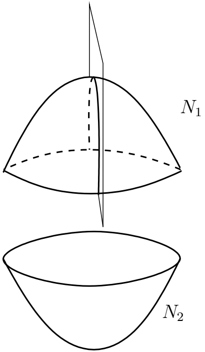
The above are given topology by seeing that $S^n \times \mathbb{R}$ parametrize affine, oriented hyperplanes.

$\circ_i$-composition is induced by grafting indexing trees.

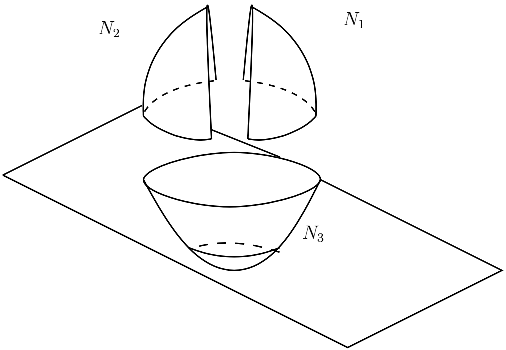
Operation in Pictures



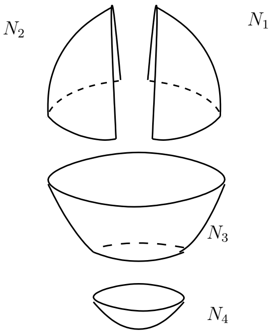
Operation in Pictures



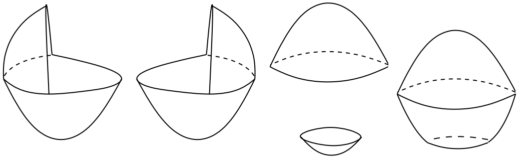
Operation in Pictures



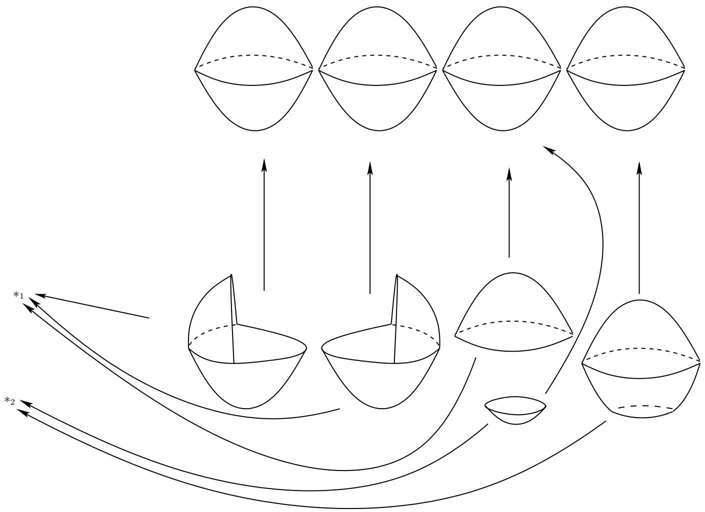
Operation in Pictures



Operation in Pictures



Operation in Pictures



Action in Symbols

Taking the functor M^{-} dualizes to the pullback-diagram

$$
\begin{array}{ccc}
M^{N}_{[T,\underline{P}]} & \longrightarrow & \left(M^{N}\right)^{k} \\
\big\downarrow & & \big\downarrow \text{res} \\
M^{\pi_0(\beta_{[T,\underline{P}]})} & \xrightarrow{\ \phi\ } & \prod_{i=1}^{k} M^{(\complement N_i)}
\end{array}
$$

.

Action in Symbols

Taking the functor M^{-} dualizes to the pullback-diagram

$$\begin{array}{ccc} M^{N}_{[T,\underline{P}]} & \longrightarrow & \left(M^{N}\right)^{k} \\ \Big\downarrow & & \Big\downarrow \text{res} \\ M^{\pi_0(\beta_{[T,\underline{P}]})} & \xrightarrow{\phi} & \prod_{i=1}^{k} M^{(\complement N_i)} \end{array}$$

We see $M^{N}_{[T,\underline{P}]} \cong \{f \in M^{N} \mid f \text{ is constant along } \beta_{[T,\underline{P}]}\}$.

Action in Symbols

Taking the functor M^{-} dualizes to the pullback-diagram

$$\begin{array}{ccc} M^N_{[T,\underline{P}]} & \longrightarrow & \left(M^N\right)^k \\ \big\downarrow & & \big\downarrow \text{res} \\ M^{\pi_0(\beta_{[T,\underline{P}]})} & \xrightarrow{\phi} & \prod_{i=1}^k M^{(\complement N_i)} \end{array}$$

We see $M^N_{[T,\underline{P}]} \cong \{f \in M^N \mid f \text{ is constant along } \beta_{[T,\underline{P}]}\}$.

So $\mathcal{Cleav}_N$ acts on M^N through the *correspondences*

$$M^N \hookleftarrow M^N_{[T,\underline{P}]} \longrightarrow \left(M^N\right)^k$$

Action in Symbols

Taking the functor M^{-} dualizes to the pullback-diagram

$$\begin{array}{ccc} M^N_{[T,\underline{P}]} & \longrightarrow & \left(M^N\right)^k \\ \big\downarrow & & \big\downarrow \text{res} \\ M^{\pi_0(\beta_{[T,\underline{P}]})} & \xrightarrow{\phi} & \prod_{i=1}^k M^{(\complement N_i)} \end{array}$$

We see $M^N_{[T,\underline{P}]} \cong \{f \in M^N \mid f \text{ is constant along } \beta_{[T,\underline{P}]}\}$.

So $\mathcal{Cleav}_N$ acts on M^N through the *correspondences*

$$M^N \hookleftarrow M^N_{[T,\underline{P}]} \longrightarrow \left(M^N\right)^k$$

Working homotopically leads us to an actual stable map

$$\mathcal{Cleav}_{S^n}(-;k) \times \left(M^{S^n}\right)^k \to M^{S^n} \wedge S^{-\dim(M)\times(k-1)}$$

Cleavage Homotopy Type

$\mathcal{D}isk_n$ represents the homotopy class of E_n-operads.

Cleavage Homotopy Type

$\mathcal{Disk}_n$ represents the homotopy class of E_n-operads.

Definition

A coloured operad $\mathcal{O}$ is called E_n if it is weakly equivalent to the trivial extension

$$\mathcal{Disk}_n \times \mathrm{Ob}(\mathcal{O})$$

Cleavage Homotopy Type

$\mathcal{Disk}_n$ represents the homotopy class of E_n-operads.

Definition

A coloured operad $\mathcal{O}$ is called E_n if it is weakly equivalent to the trivial extension

$$\mathcal{Disk}_n \times \mathrm{Ob}(\mathcal{O})$$

Theorem (B.)

The coloured operad $\mathcal{Cleav}_{S^n}$ *is* E_{n+1}

Cleavage Homotopy Type

$\mathcal{D}isk_n$ represents the homotopy class of E_n-operads.

Definition

A coloured operad $\mathcal{O}$ is called E_n if it is weakly equivalent to the trivial extension

$$\mathcal{D}isk_n \times \mathrm{Ob}(\mathcal{O})$$

Theorem (B.)

The coloured operad $\mathcal{C}leav_{S^n}$ is E_{n+1}

$\mathcal{C}leav_{S^n}$ can be twisted by $\mathrm{SO}(n+1)$, this leads to

Cleavage Homotopy Type

$\mathcal{Disk}_n$ represents the homotopy class of E_n-operads.

Definition

A coloured operad $\mathcal{O}$ is called E_n if it is weakly equivalent to the trivial extension

$$\mathcal{Disk}_n \times \mathrm{Ob}(\mathcal{O})$$

Theorem (B.)

The coloured operad $\mathcal{Cleav}_{S^n}$ is E_{n+1}

$\mathcal{Cleav}_{S^n}$ can be twisted by $\mathrm{SO}(n+1)$, this leads to

Corollary

$\mathbb{H}_*\left(M^{S^n}\right)$ *is a* $(n+1)$*-Batalin-Vilkovisky-algebra*

Group Actions

For a group G acting freely on $S^n \subseteq \mathbb{R}^{n+1}$, The group action leads to an algebra structure on $\mathrm{H}_*^G(M^{S^n})$.

Group Actions

For a group G acting freely on $S^n \subseteq \mathbb{R}^{n+1}$, The group action leads to an algebra structure on $\mathrm{H}_*^G(M^{S^n})$.

The inclusions of unit spheres

$$S^1 \hookrightarrow S^2 \hookrightarrow \cdots$$

Group Actions

For a group G acting freely on $S^n \subseteq \mathbb{R}^{n+1}$, The group action leads to an algebra structure on $\mathrm{H}_*^G(M^{S^n})$.

The inclusions of unit spheres

$$S^1 \hookrightarrow S^2 \hookrightarrow \cdots$$

gives an inclusion of operads

$$\mathcal{C}leav_{S^1} \hookrightarrow \mathcal{C}leav_{S^2} \hookrightarrow \cdots$$

Group Actions

For a group G acting freely on $S^n \subseteq \mathbb{R}^{n+1}$, The group action leads to an algebra structure on $\mathrm{H}_*^G(M^{S^n})$.

The inclusions of unit spheres

$$S^1 \hookrightarrow S^2 \hookrightarrow \cdots$$

gives an inclusion of operads

$$\mathcal{C}leav_{S^1} \hookrightarrow \mathcal{C}leav_{S^2} \hookrightarrow \cdots$$

and the limiting action gives a E_∞-structure on $M^{S^\infty} \wedge S^{-\dim(M)} \simeq M \wedge S^{-\dim(M)}$.

Group Actions

For a group G acting freely on $S^n \subseteq \mathbb{R}^{n+1}$, The group action leads to an algebra structure on $\mathrm{H}_*^G(M^{S^n})$.

The inclusions of unit spheres

$$S^1 \hookrightarrow S^2 \hookrightarrow \cdots$$

gives an inclusion of operads

$$\mathcal{C}\mathit{leav}_{S^1} \hookrightarrow \mathcal{C}\mathit{leav}_{S^2} \hookrightarrow \cdots$$

and the limiting action gives a E_∞-structure on $M^{S^\infty} \wedge S^{-\dim(M)} \simeq M \wedge S^{-\dim(M)}$.

This is a spectrum-level definition of the intersection product of manifolds,

Group Actions

For a group G acting freely on $S^n \subseteq \mathbb{R}^{n+1}$, The group action leads to an algebra structure on $\mathrm{H}_*^G(M^{S^n})$.

The inclusions of unit spheres

$$S^1 \hookrightarrow S^2 \hookrightarrow \cdots$$

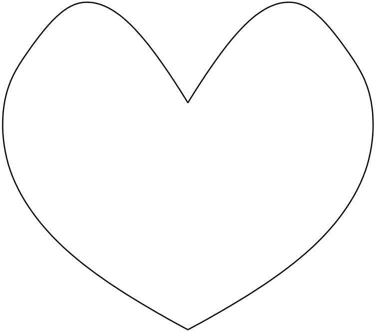
gives an inclusion of operads

$$\mathcal{C}leav_{S^1} \hookrightarrow \mathcal{C}leav_{S^2} \hookrightarrow \cdots$$

and the limiting action gives a E_∞-structure on $M^{S^\infty} \wedge S^{-\dim(M)} \simeq M \wedge S^{-\dim(M)}$.

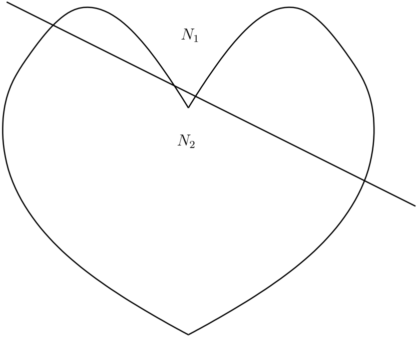
This is a spectrum-level definition of the intersection product of manifolds, and gives similar statements for $\mathrm{H}_*^G(M)$ where G acts freely on $S^\infty \subset \mathbb{R}^\infty$.

Interplay with Khovanov Homology



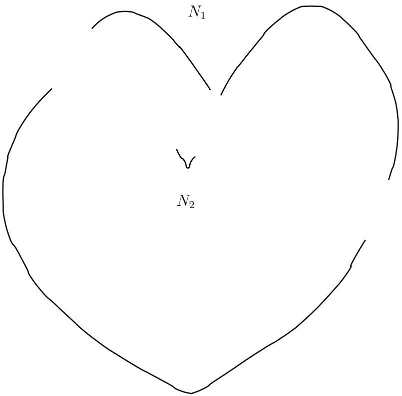
Take an embedding $K\colon S^n \hookrightarrow \mathbb{R}^N$.

Interplay with Khovanov Homology



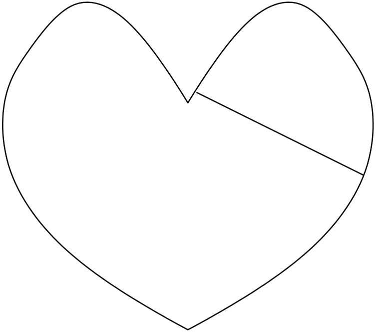
Take an embedding $K: S^n \hookrightarrow \mathbb{R}^N$.This gives a new operad $\mathcal{C}leav_K$

Interplay with Khovanov Homology



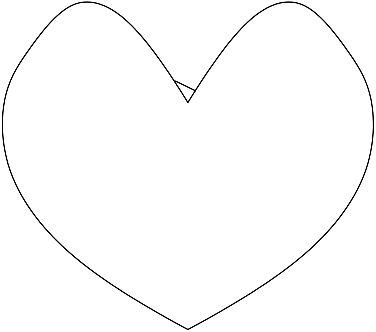
However, the action on M^{S^1} differs from that of $\mathcal{C}leav_{S^1}$

Interplay with Khovanov Homology



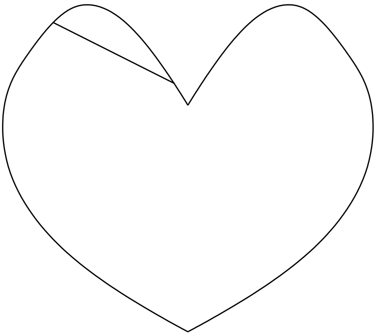
In the String Topology action this can be seen as a product...

Interplay with Khovanov Homology



... followed by a coproduct...

Interplay with Khovanov Homology



... followed by a product

Khovanov Homology and TQFTs

Theorem (Tamanoi ’08)

In String Topology, a product followed by a (graded) coproduct is trivial

Khovanov Homology and TQFTs

Theorem (Tamanoi '08)

In String Topology, a product followed by a (graded) coproduct is trivial

In particular, a TQFT-construction of Cohen-Godin has most associated operations trivial.

Khovanov Homology and TQFTs

Theorem (Tamanoi '08)

In String Topology, a product followed by a (graded) coproduct is trivial

In particular, a TQFT-construction of Cohen-Godin has most associated operations trivial.

For $\mathcal{C}leav_K$ this means that no new operations arise in $\mathbb{H}_*(M^{S^1})$ from isotoping $K(S^1) \subset \mathbb{R}^2$ away from the unit circle.

Khovanov Homology and TQFTs

Theorem (Tamanoi '08)

In String Topology, a product followed by a (graded) coproduct is trivial

In particular, a TQFT-construction of Cohen-Godin has most associated operations trivial.

For $\mathcal{Cleav}_K$ this means that no new operations arise in $\mathbb{H}_*(M^{S^1})$ from isotoping $K(S^1) \subset \mathbb{R}^2$ away from the unit circle.

For a knot $K\colon S^1 \to \mathbb{R}^3$ the gordian knot operad $\mathcal{Gord}_K$ is constructed with only slight modifications from $\mathcal{Cleav}_K$

Khovanov Homology and TQFTs

Theorem (Tamanoi '08)

In String Topology, a product followed by a (graded) coproduct is trivial

In particular, a TQFT-construction of Cohen-Godin has most associated operations trivial.

For $\mathcal{C}leav_K$ this means that no new operations arise in $\mathbb{H}_*(M^{S^1})$ from isotoping $K(S^1) \subset \mathbb{R}^2$ away from the unit circle.

For a knot $K\colon S^1 \to \mathbb{R}^3$ the gordian knot operad $\mathcal{G}ord_K$ is constructed with only slight modifications from $\mathcal{C}leav_K$

Theorem (b. (in progress))

$\mathrm{H}_*(\mathcal{G}ord_K)$ *acts on* $\mathbb{H}_*(M^{\coprod_{\mathbb{N}} S^1})$ *to produce a knot invariant*

Khovanov Homology and TQFTs

Theorem (Tamanoi '08)

In String Topology, a product followed by a (graded) coproduct is trivial

In particular, a TQFT-construction of Cohen-Godin has most associated operations trivial.

For $\mathcal{C}leav_K$ this means that no new operations arise in $\mathbb{H}_*(M^{S^1})$ from isotoping $K(S^1) \subset \mathbb{R}^2$ away from the unit circle.

For a knot $K\colon S^1 \to \mathbb{R}^3$ the gordian knot operad $\mathcal{G}ord_K$ is constructed with only slight modifications from $\mathcal{C}leav_K$

Theorem (b. (in progress))

$\mathrm{H}_*(\mathcal{G}ord_K)$ *acts on* $\mathbb{H}_*(M^{\coprod_{\mathbb{N}} S^1})$ *to produce a knot invariant*

This is a Khovanov homology construction, and a different flavour of TQFT than Cohen-Godin.

The knot invariant action comes from a 2-categorical correspondence structure combining string topology with cobordisms $O(v) \hookrightarrow C_{v,w} \hookleftarrow O(w)$

The knot invariant action comes from a 2-categorical correspondence structure combining string topology with cobordisms $O(v) \hookrightarrow C_{v,w} \hookleftarrow O(w)$

$$\begin{array}{ccccc}
M^{O(v)} & \longleftarrow & M^{O(v)}_{[T,\underline{P}]_v} & \longrightarrow & \left(M^{O(v)}\right)^k \\
\uparrow & & \uparrow & & \downarrow \\
M^{C_{v,w}} & \longleftarrow & M^{C_{v,w}}_{[T,\underline{P}]} & \longrightarrow & \left(M^{C_{v,w}}\right)^k \\
\downarrow & & \downarrow & & \uparrow \\
M^{O(w)} & \longleftarrow & M^{O(w)}_{[T,\underline{P}]_w} & \longrightarrow & \left(M^{O(w)}\right)^k
\end{array}$$