An Introduction to the Stolz-Teichner Program

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Outline of Talk

Classical Topological Field Theories

Motivating Principles

Graeme Segal [3] had an idea for describing conformal field theory.

- Let $E \to M$ be a vector bundle with connection ∇.
- Make association to points: for $x \in M$ we associate

$$x \longmapsto E_x$$

- For (piecewise smooth) paths ${\gamma_x}^y$ between points x and y we associate the parallel transport map,

$${\gamma_x}^y \longmapsto {\tau_x}^y : E_x \xrightarrow{\simeq} E_y .$$

- In this way, a *1-dimensional field theory* over M may be viewed as a vector bundle with connection.

Classical Topological Field Theories

Motivating Principles

- Segal suggested $2d$-CFT over M are cocycles for some elliptic cohomology theory over M.
- The CFT is given by:

$$\text{Loops} \longmapsto \text{Hilbert Space}$$

and

$$\text{Conformal Surfaces} \longmapsto \text{Hilbert-Schmidt Operators}$$

- **Problem:** Excision does not hold

Classical Topological Field Theories

Motivating Principles

Some results and conjectures to date in the Stolz-Teichner Program:

- 0-field theories over $M \cong C^\infty(M)$
- $0|1$-field theories over $M \cong \Omega^{\mathrm{ev}}_{\mathrm{cl}}(M)$
- 1-field theories over $M \cong \mathrm{VB}^\nabla(M)$
- $1|1$-field theories over $M \cong K(M)$
- $2|1$-field theories over $M \cong \mathrm{TMF}^*(M)$ (conjectured)

Classical Topological Field Theories

Motivating Principles

- Atiyah started this topological field theory business off [1].
- Consider the following categories:

Definition

The d-dimensional bordism category, d-B, consisting of

- **Objects:** *Closed $(d-1)$-manifolds (orientable)*
- **Morphisms:** *d-bordisms (oriented)*

Definition

The category Vect *of vector spaces*

- **Objects:** *Vector spaces (finite dimensional)*
- **Morphisms:** *Linear transformations*

Definition

A d-dimensional topological field theory is a (symmetric monoidal) functor

- Basically:

$$(d-1)\text{-dimensional manifolds} \longrightarrow \text{Vector Spaces}$$

and

$$d\text{-dimensional bordisms} \longrightarrow \text{Linear Transformations}$$

Classical Topological Field Theories

Motivating Principles

- Symbolically, the *category* of d-dimensional TFTs is

$$d\text{-TFT} := \mathrm{Fun}^{\otimes}(d\text{-B}, \mathrm{Vect}_{\mathbb{R}}).$$

  Here "$\otimes$" emphasises symmetric monoidal functors.
- Suffices to consider closed, connected objects $\Longrightarrow$ omit reference to symmetric monoidal structure.
- In the zero-dimensional case:
  - Objects in 0-B is just $\emptyset$
  - Symmetric monoidal $\Longrightarrow$ unit $\to$ unit, so

$$\emptyset \longmapsto \mathbb{R}.$$

  - Morphisms in 0-B are points, thought of a bordisms $\emptyset \to \emptyset$. Thus

$$\mathrm{pt} \longmapsto \lambda \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}, \mathbb{R}) = \mathbb{R}^{\times}.$$

Classical Topological Field Theories

Motivating Principles

- Think of $\mathbb{R}$ as a discrete category
  - Objects are real numbers λ.
  - Morphisms are identity morphisms.
- Thus 0-TFT $\cong \mathbb{R}$ as discrete categories.
  - Given by $E \mapsto E(\mathrm{pt})$
- Objective is to obtain invariants of a space X by "parametrising" 0-TFT by X.

Definition

Define d-TFT(X) as functors d-B$(X) \to$ Vect. Here d-B(X) is just d-B where objects and bordisms are equipped with a smooth map to X.

- Bordisms are defined up to diffeomorphism $\implies$ map to X is diffeomorphism invariant,
- This is where all the invariant information about X is coming from.

Classical Topological Field Theories

Motivating Principles

- In zero-dimensions: 0-B(X) consists of
  - Objects: $\emptyset \to X$
  - Morphisms: pt $\hookrightarrow X$.
  - As discrete categories: 0-B(X) $\cong X$.
- Then

$$\begin{aligned} \text{0-TFT}(X) &= \text{Fun}^{\otimes}(\text{0-B}(X), \text{Vect}) \\ &\cong \text{Fun}(X, \mathbb{R}) \\ &= \mathrm{Maps}(X, \mathbb{R}). \end{aligned}$$

- There is no smoothness condition on these maps!
- Hohnhold et. al., rectify this by adding smoothness and generalise by adding supersymmetry [2].
- Idea is to consider *families* of objects.

Preliminary Category Theory

Fibered category theory:

- Generalise the notion of a bundle over a space to a "category-theoretic" bundle over a category
- See [5] for more fibered category theory.

Definition

A functor $p : \mathsf{B} \to \mathsf{S}$ *is a fibration if pullbacks exist for every object in* S.

- For example: the following functor is a fibration,

$$\mathfrak{F} : \text{Vector bundles} \longmapsto \text{Base space}$$

Preliminary Category Theory

Definition

Say a category B *is fibered over a category* S *if there exists a fibration* $p : \mathrm{B} \to \mathrm{S}$.

- For $s \in \mathrm{S}$, the fiber B_s over S is the subcategory

$$B_s = \{(b, f) \in B^0 \times B^1 : (b, f) \mapsto (s, \mathrm{id}_s)\}$$

  where B^0 are objects and B^1 are morphisms.
- e.g. VB $\to$ Man is fibration
  - For $S \in$ Man the fiber $\mathrm{VB}_S = \{\text{bundles over } S\}$
- A *fibered functor* $\mathfrak{F}$, between categories B and V, fibered over S, is a functor $\mathrm{B} \to \mathrm{V}$ that preserves the fibration.

Preliminary Category Theory

- Denote by

$$\mathrm{Fun}_{\mathsf{S}}(\mathsf{B}, \mathsf{V})$$

  the category of fibered functors between fibered categories B and V over S.
  - **Objects:** fibered functors
  - **Morphisms:** (fibered) natural transformations.

Theorem

For a functor $\mathfrak{F} : \mathsf{S}^{\mathrm{op}} \to \mathsf{Set}$ *there exists a corresponding fibration* $\underline{\mathfrak{F}} \to \mathsf{S}$. □

- Proof is by construction,

$$\underline{\mathfrak{F}} = \mathsf{S} \times \mathsf{Set} = \{(s, \mathfrak{F}(s)) \text{ and morphisms}\}.$$

  Fibration $\underline{\mathfrak{F}} \to \mathsf{S}$ is given by the forgetful map.

Preliminary Category Theory

- **Key Point:** For an object $s \in \mathsf{S}$ there exists a natural fibration $p : \underline{s} \to \mathsf{S}$.
- Given by thinking of s in terms of its functor of points $s \equiv \mathsf{S}(-, s)$. Then apply theorem.
- Explicitly $\underline{s} = \mathsf{S} \times \mathsf{S}(-, s)$ and the forgetful map $\underline{s} \to \mathsf{S}$ is a fibration.
  - Fiber is $\underline{s}_t = \mathsf{S}(t, s)$ for each $t \in \mathsf{S}$.
- **Motivation:** If $\mathsf{V} \to \mathsf{S}$ is a fibration, then

$$\mathrm{Fun}_{\mathsf{S}}(\underline{s}, \mathsf{V}) \cong \mathsf{V}_s.$$

- In particular, if $\mathsf{V} = \underline{s}'$, for $s' \in \mathsf{S}$, then

$$\mathrm{Fun}_{\mathsf{S}}(\underline{s}, \underline{s}') \cong \underline{s}'_s = \mathsf{S}(s, s').$$

Definition

We define the d-dimensional family bordism category over M, $d\text{-B}^f(M)$, as a category with, for each manifold S,

- **Objects:** *d-dimensional fiber bundles $Y \to S$ with closed, connected fiber and smooth map $Y \to M$.*
- **Morphisms:** *bundle morphisms that are fiber-wise diffeomorphisms.*

Definition

A smooth d-dimensional TFT *over a manifold M is a fibered functor from $d\text{-B}^f(M)$ to $\mathbb{R}$,*

$$d\text{-TFT}(M) := \text{Fun}_{\text{Man}}(d\text{-B}^f(M), \mathbb{R}),$$

where Man *is the category of smooth manifolds.*

Proposition

There exists a bijection

$$\text{0-TFT}(M) \cong C^\infty(M).$$

Proof.

Note that $\text{0-B}^f(M) \cong \underline{M}$ since, for each S,

$$\text{0-B}^f(M) : S \longmapsto (S \to M) = \underline{M}(S).$$

Then for: $\text{0-TFT}(M) = \text{Fun}_{\text{Man}}(\text{0-B}^f(M), \underline{\mathbb{R}})$,

$$\text{0-TFT}(M) \cong \text{Fun}_{\text{Man}}(\underline{M}, \underline{\mathbb{R}}) \cong \text{Man}(M, \mathbb{R}) = C^\infty(M).$$

□

For supersymmetric field theories:

- Generalises straightforwardly.
- Need to fiber over the category of supermanifolds.
- Defined for dimension $0|\delta$.

Definition

The $0|\delta$*-dimensional family bordism category* $0|\delta\text{-B}^f(M)$ *is defined just as non-SUSY case, except bundles have* $0|\delta$*-dimensional fiber.*

- **Theorem:** (WLOG) It suffices to consider product bundles of the form $S \times \mathbb{R}^{0|\delta} \to S$ as objects.

Smooth and SUSY Field Theories

SUSY Field Theories

Definition

The category of $0|\delta$*-dimensional* TFTs *over* M *is defined as the following fibered functor category,*

$$0|\delta\text{-TFT}(M) := \text{Fun}_{\text{SM}}(0|\delta\text{-B}^f(M), \mathbb{R}),$$

where SM *is the category of supermanifolds.*

Theorem

There is a bijection,

$$0|1\text{-TFT}(M) \cong \Omega^0_{\text{cl}}(M).$$

- To explain this result we need a better understanding of $0|1\text{-B}^f(M)$

Preliminary Theory

The Functor of Points Approach

- For a supermanifold $M \in \mathrm{SM}$, we can think of it as

$$S \longmapsto M(S) := \mathrm{SM}(S, M).$$

where S varies over all supermanifolds.
- Note $\mathrm{SM}(-, M)$ varies functorially (and contravariantly) for all $S \in \mathrm{SM}$, i.e., for $f : T \to S$,

$$\mathrm{SM}(f, M) : \mathrm{SM}(S, M) \longrightarrow \mathrm{SM}(T, M)$$

given by $\phi \mapsto \phi \circ f$.
- Call $\mathrm{SM}(-, M)$ the *functor of points* of M.
- **Advantage:** There exists a result to the effect (see [2]),

$$\mathrm{SM}(S, M) \cong \mathrm{Alg}(C^\infty(M), C^\infty(S))$$

$\Longrightarrow$ Suffices to look at algebra homomorphisms between smooth functions!

Preliminary Theory

Generalised Supermanifolds

- In the functor of points approach we consider embedding

$$\mathsf{SM} \longrightarrow \mathsf{Fun}(\mathsf{SM}^{\mathrm{op}}, \mathsf{Set})$$

given by,

$$M \longmapsto (S \longmapsto \mathsf{SM}(S, M)).$$

- This embedding is known as a *Yoneda embedding*.
- Call a functor $\mathfrak{F} \in \mathsf{Fun}(\mathsf{SM}^{\mathrm{op}}, \mathsf{Set})$ a *generalised supermanifold*.
- Generalised supermanifolds in the image of the embedding are representable.
  - i.e., $\mathfrak{F}$ is representable if it is naturally isomorphic to the functor of points of some $N \in \mathsf{SM}$.

Preliminary Theory

Generalised Supermanifolds

- Denote by $\underline{\mathrm{SM}}(-, M)$ the *inner Hom*
- For $S, T \in \mathrm{SM}$ the inner Hom satisfies,

$$\underline{\mathrm{SM}}(T, M)(S) = \mathrm{SM}(S \times T, M).$$

- Thus $\underline{\mathrm{SM}}(T, M)$ is a *generalised supermanifold*
- As it turns out $\underline{\mathrm{SM}}(\mathbb{R}^{0|n}, M)$ is representable!
  - i.e., $\underline{\mathrm{SM}}(\mathbb{R}^{0|n}, M) \cong \mathrm{SM}(-, N)$ for some $N \in \mathrm{SM}$.

Preliminary Theory

Generalised Supermanifolds

- For $M \in \mathrm{SM}$, the *odd tangent bundle* ΠTM of M is constructed by reversing parity of the fibers of TM.
- **Key point**:
$$C^\infty(\Pi TM) \cong \Omega^\bullet(M).$$
- Local coordinates on ΠTM are $(x^i, \mathrm{d}x^i)$, where x^i are local coordinates on M.
- Moreover,
$$C^\infty(\Pi TM)^{\mathrm{ev}} = C^\infty(\Pi TM, \mathbb{R}) \cong \Omega^{\mathrm{ev}}(M)$$
and
$$C^\infty(\Pi TM)^{\mathrm{odd}} = C^\infty(\Pi TM, \mathbb{R}^{0|1}) \cong \Omega^{\mathrm{odd}}(M).$$

Preliminary Theory

Generalised Supermanifolds

Proposition

For any $M \in \mathsf{SM}$ *there exists an isomorphism as generalised supermanifolds*

$$\left(T \longmapsto \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, M)(T)\right) \cong (T \longmapsto \mathsf{SM}(T, \Pi TM))\,.$$

□

- i.e., $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, M) \cong \Pi TM$
- More generally, it follows by induction that

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, M) \cong (\Pi T)^{\delta} M$$

Preliminary Theory
Group Actions

- The super-point $\mathbb{R}^{0|1}$ admits a super-Lie group structure:

$$\mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \longrightarrow \mathbb{R}^{0|1} \quad \text{given by} \quad (\eta, \theta) \longmapsto \eta + \theta$$

- and a dilational action

$$\mathbb{R}^{\times} \times \mathbb{R}^{0|1} \longrightarrow \mathbb{R}^{0|1} \quad \text{given by} \quad (\lambda, \theta) \longmapsto \lambda\theta$$

Preliminary Theory

Group Actions

Definition

The inner Diffeomorphism group of the super-point $\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})$ *is the super-Lie group-valued functor*

$$S \longmapsto \mathrm{Diff}_S(S \times \mathbb{R}^{0|1}, S \times \mathbb{R}^{0|1}).$$

The S-subscript denotes compatibility with projection onto S.

- Diagramatically $\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})$ defines, for each S, the following commutative diagram

$$\begin{array}{ccc} S \times \mathbb{R}^{0|1} & \longrightarrow & S \times \mathbb{R}^{0|1} \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

Preliminary Theory

Group Actions

Lemma

There exists an isomorphism of generalised super-Lie groups,

$$\underline{\mathrm{Diff}}(\mathbb{R}^{0|1}) \cong \mathbb{R}^{\times} \ltimes \mathbb{R}^{0|1}$$

- This means we may think of elements of $\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})$ as pairs $(\lambda, \theta) \in \mathbb{R}^{1|1}$.
- We have a diffeomorphism action on the super-point:

$$\mathbb{R}^{0|1} \times (\mathbb{R}^{\times} \ltimes \mathbb{R}^{0|1}) \longrightarrow \mathbb{R}^{0|1}$$

given by

$$(\eta, (\lambda, \theta)) \longmapsto (\eta\lambda, \theta) \longmapsto \eta\lambda + \theta.$$

Preliminary Theory

Group Actions

- Diffeomorphism action on $\mathbb{R}^{0|1}$ extends to an action on ΠTM
- In local coordinates $(x^i, \mathrm{d}x^i)$ on ΠTM group action is

$$\mu : \underline{\mathrm{Diff}}(\mathbb{R}^{0|1}) \times \Pi TM \longrightarrow \Pi TM$$

given by

$$((\lambda, \theta), (x^i, \mathrm{d}x^i)) \stackrel{\mu}{\longmapsto} (x^i + \theta \mathrm{d}x^i, \lambda \mathrm{d}x^i).$$

- On functions $\mu^*_{\lambda,\theta} : \Omega^\bullet(M) \to \Omega^\bullet(M)[\theta]$ is given by

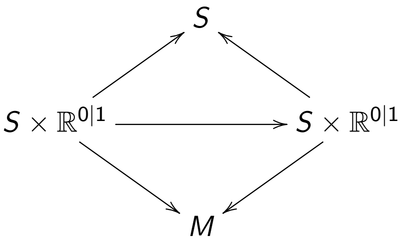
$$\Omega^k(M) \ni \omega \longmapsto \mu^*_{\lambda,\theta}(\omega) = \lambda^k(\omega + \theta \mathrm{d}\omega).$$

Proof Of Theorem

- **Theorem:** $0|1\text{-TFT}(M) \cong \Omega^0_{\mathrm{cl}}(M)$.
- Recall that

$$0|1\text{-TFT}(M) := \mathrm{Fun}_{\mathrm{SM}}(0|1\text{-B}^f(M), \mathbb{R}).$$

- **Claim:** $0|1\text{-B}^f(M)$ is just the fixed point set of the diffeomorphism action on ΠTM.
- For each $S \in \mathrm{SM}$ we have commutative diagram



The $0|1$-Dimensional Case

- Commutativity implies $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, M) \cong \Pi TM$ is invariant under diffeomorphism action.
- Hence
$$0|1\text{-B}^f(M) \cong \Pi TM / \underline{\mathrm{Diff}}(\mathbb{R}^{0|1}).$$
This is an equivalence as fibered categories.
- Then we find

$$\begin{aligned} 0|1\text{-TFT}(M) &:= \mathrm{Fun}_{\mathsf{SM}}(0|1\text{-B}^f(M), \underline{\mathbb{R}}) \\ &\cong C^\infty(\Pi TM)^{\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})} \\ &\cong \{\omega \in \Omega^\bullet(M) \colon \mu^*_{\lambda,\theta}\omega = \omega\} \\ &= \Omega^0_{\mathrm{cl}}(M). \end{aligned}$$

□

Twisted Field Theories

- It is natural to ask whether we can obtain higher degree de Rham cocycles.
- We can indeed! Recall the diffeomorphism action

$$\mu : \underline{\mathsf{Diff}}(\mathbb{R}^{0|1}) \times \Pi TM \to \Pi TM$$

which, on functions, is of the form

$$\Omega^k(M) \ni \omega \longmapsto \mu^*_{\lambda,\theta}(\omega) = \lambda^k(\omega + \theta \mathrm{d}\omega)$$

- To obtain the higher degree cocycles, we need to pick out the multiplicative action.

Twisted Field Theories

- Consider the group homomorphism

$$\rho : \underline{\mathrm{Diff}}(\mathbb{R}^{0|1}) \longrightarrow \mathbb{R}^{\times}$$

- We construct, to this homomorphism, the associated line bundle over ΠTM,

$$L := \Pi TM \times_{\rho} \mathbb{R}.$$

- It defined by the equivalence relation

$$(g \cdot \zeta, t) \sim (\zeta, \rho(g)^{-1} t)$$

for $g \in \underline{\mathrm{Diff}}(\mathbb{R}^{0|1}), \zeta \in \Pi TM$ and $t \in \mathbb{R}$.

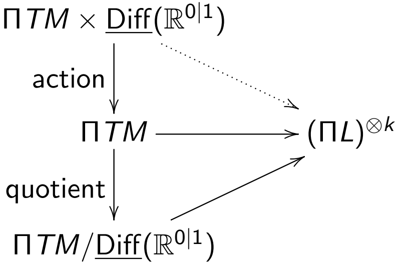
Twisted Field Theories

Definition

A $0|1$-*field theory of degree k over M, denoted* $0|1\text{-TFT}^k(M)$, *is defined as,*

$$0|1\text{-TFT}^k(M) := \Gamma(0|1\text{-B}^f(M), (\Pi L)^{\otimes k}).$$

- Recall $0|1\text{-B}^f(M) \cong \Pi TM/\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})$.
- $0|1\text{-B}^f(M) \to (\Pi L)^{\otimes k}$ are defined by commutativity,



Twisted Field Theories

- For $k = 0$ we find

$$\begin{aligned} 0|1\text{-TFT}^0(M) &\cong C^\infty(\Pi TM)^{\underline{\text{Diff}}(\mathbb{R}^{0|1})} \\ &\cong \Omega^0_{\text{cl}}(M) \cong 0|1\text{-TFT}(M). \end{aligned}$$

  As such $0|1$-TFT(M) is by definition untwisted.
- Parity of L is reversed since $\Omega^{k=\text{odd}}(M)$ may be described as a vector space with odd variables.

Proposition

We have bijection

$$\Gamma(0|1\text{-B}^f(M), (\Pi L)^{\otimes k}) \cong \{\mu_g^*(\omega) = \rho(g)^k \omega\}$$

for $g \in \underline{\text{Diff}}(\mathbb{R}^{0|1})$ *and* $\omega \in C^\infty(\Pi TM) \cong \Omega^\bullet(M)$.

Twisted Field Theories

- Recall action

$$\mu : \underline{\mathsf{Diff}}(\mathbb{R}^{0|1}) \times \Pi TM \to \Pi TM$$

which, on functions, is of the form

$$\Omega^k(M) \ni \omega \longmapsto \mu^*_{\lambda,\theta}(\omega) = \lambda^k(\omega + \theta \mathrm{d}\omega).$$

- Then $\mu^*_{\lambda,\theta}\omega = \lambda^k\omega \iff \omega \in \Omega^k(M)$ and $\mathrm{d}\omega = 0$.
- Thus,

$$0|1\text{-TFT}^k(M) \cong \Omega^k_{\mathrm{cl}}(M).$$

Generalisations

Euclidean Field Theories

- Can generalise to Euclidean field theories
- **Motivation:** Put Riemannian structure on the bundles in the Bordism category
  - specialise to flat Riemannian metrics.
- Consider subgroup $\underline{\mathrm{Iso}}(\mathbb{R}^{0|1}) \leq \underline{\mathrm{Diff}}(\mathbb{R}^{0|1})$ consisting of reflections and odd translations, i.e.,

$$\underline{\mathrm{Iso}}(\mathbb{R}^{0|1}) = \{\pm 1\} \ltimes \mathbb{R}^{0|1}$$

- We have group action $\underline{\mathrm{Iso}}(\mathbb{R}^{0|1})$ on ΠTM, given by

$$\Omega^k(M) \longrightarrow \Omega^\bullet(M)[\theta] \quad \text{with} \quad \omega \longmapsto (\pm 1)^k(\omega + \theta \mathrm{d}\omega)$$

- Clearly $\Omega^{\mathrm{ev}}_{\mathrm{cl}}(M)$ is invariant under this action.

Generalisations

Euclidean Field Theories

- We define the $0|1$-*Euclidean bordism family category* over M, $0|1\text{-EB}(M)$, just as $0|1\text{-B}^f(M)$, replacing $\underline{\text{Diff}}(\mathbb{R}^{0|1})$ by $\underline{\text{Iso}}(\mathbb{R}^{0|1})$.
- We analogously have

$$0|1\text{-EB}(M) \cong \Pi TM/\underline{\text{Iso}}(\mathbb{R}^{0|1})$$

- We define $0|1\text{-EFT}(M)$ in the obvious way,

$$0|1\text{-EFT}(M) := \text{Fun}_{\text{SM}}(0|1\text{-EB}(M), \underline{\mathbb{R}}).$$

Theorem

There exists a bijection

$$0|1\text{-EFT}(M) \cong \Omega^{\text{ev}}_{\text{cl}}(M)$$

Generalisations

Euclidean Field Theories

- Defining twisted Euclidean field theories in the same way as for ordinary field theories yields the following theorem.

Theorem

There is a bijection

$$0|1\text{-EFT}^1(M) \cong \Omega_{\mathrm{cl}}^{\mathrm{odd}}(M).$$

□

- More generally

$$0|1\text{-EFT}^{\mathrm{ev/odd}}(M) \cong \Omega_{\mathrm{cl}}^{\mathrm{ev/odd}}(M).$$

Generalisations

Twisted Cohomology

- Let $V \to M$ be a vector bundle with a flat connection ∇ (i.e. $\nabla^2 = 0$).
- Denote V-valued differential forms by

$$\Omega^\bullet(M, V) := \Gamma(M, V) \otimes_{C^\infty(M)} \Omega^\bullet(M).$$

- Then $\nabla : \Omega^0(M, V) \to \Omega^1(M, V)$.
- Write $\mathrm{d}_k^\nabla : \Omega^k(M, V) \to \Omega^{k+1}(M, V)$.
- ∇ flat $\implies (\mathrm{d}_k^\nabla)^2 = 0$.
- The pair $(\Omega^\bullet(M, V), \mathrm{d}^\nabla) = \Omega^\bullet_\nabla(M, V)$ forms a differential complex.
- We define the k-th *twisted de Rham cohomology group* by

$$H^k(M, V) = \ker \mathrm{d}_k^\nabla / \mathrm{im}\ \mathrm{d}_{k-1}^\nabla$$

Generalisations

Twisted Cohomology

- **Motivation:** A V-valued field theory reduces to the ordinary field theory if V is the trivial line bundle.
- **Observation:**

$$\Omega^\bullet_\nabla(M, V) = \Gamma(\Pi TM, \pi^*(V, \nabla))$$

where $\pi : \Pi TM \to M$ and $\pi^* V \to \Pi TM$ is bundle with connection $\pi^*\nabla$.

- If $V = M \times \mathbb{R}$, then

$$\Gamma(\Pi TM, \pi^* V) = C^\infty(\Pi TM) \cong \Omega^\bullet(M)$$

and $\pi^*\nabla = \mathrm{d}$.

Generalisations

Twisted Cohomology

- **Recall:** for each S a $0|1$-field theory is defined on objects in the bordism category,

$$S \times \mathbb{R}^{0|1} \longrightarrow S \qquad \text{and} \qquad \phi \in \mathrm{SM}(S, \Pi TM).$$

- Consider the pullback bundle $\pi^* V$
- For each $\phi \in \mathrm{SM}(S, \Pi TM)$ we have

$$\pi^* V_\phi = (\pi \circ \phi)^* V.$$

- Want to construct fibered category

$$\underline{\pi^* V} \longrightarrow \mathrm{SM}$$

with fiber

$$\underline{\pi^* V}_S = \{\Gamma(S, (\pi \circ \phi)^* V) \colon \phi \in \underline{\Pi TM}_S\}$$

Generalisations

Twisted Cohomology

- **Claim:** $\underline{\pi^* V}_{\Pi TM} \cong \Gamma(\Pi TM, \pi^* V)$.
- Note that

$$\underline{\pi^* V}_{\Pi TM} = \{\Gamma(\Pi TM, (\pi \circ \phi)^* V) \colon \phi \in \underline{\Pi TM}_{\Pi TM}\}$$

- For $\phi \in \underline{\Pi TM}_{\Pi TM}$ it is cartesian $\iff$ it is an isomorphism
- If V is equipped with connection ∇ then $\pi^*\nabla$ is connection on $\pi^* V$ and

$$\underline{\pi^* V}_{\Pi TM} \cong \Gamma(\Pi TM, \pi^*(V, \nabla)) = \Omega^\bullet_\nabla(M, V).$$

Definition

Define a V-valued $0|1$-dimensional field theory over M with (flat) connection ∇ by

$$0|1\text{-TFT}_{\nabla}(M,V) := \mathsf{Fun}_{\mathsf{SM}}(0|1\text{-B}^f(M), \underline{\pi^*(V,\nabla)})$$

- Then

$$0|1\text{-TFT}_{\nabla}(M,V) \cong \Gamma(\Pi TM, \pi^*(V,\nabla))^{\underline{\mathsf{Diff}}(\mathbb{R}^{0|1})} = \Omega^{\bullet}_{\nabla}(M,V)^{\underline{\mathsf{Diff}}(\mathbb{R}^{0|1})}.$$

Theorem

There exists a bijection, $0|1\text{-TFT}_{\nabla}(M,V) \cong \ker \nabla$.

- It remains to find $\underline{\mathsf{Diff}}(\mathbb{R}^{0|1})$-action on sections.

Generalisations

Twisted Cohomology

- Let $\xi \in \Gamma(M, V)$. Note that

$$\mathbb{R}^{0|1} \xrightarrow{\Phi} M \xrightarrow{\xi} V.$$

Thus $\xi : \Pi TM \to \Pi TV$.

- In this way $\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})$ action extends to action on sections i.e.,

$$\mu_{\lambda,\theta} : \Pi TM \to \Pi TM \quad \text{with} \quad \mu^*_{\lambda,\theta}(\xi) := \xi \circ \mu_{\lambda,\theta}$$

- The generating vector field for $\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})$-action is d.
- This lifts to an action on sections iff ∇ is flat.

$$\mu^*_{\lambda,\theta}(\xi) = \xi + \theta \nabla \xi.$$

- e.g. if $V \to M$ is trivial line bundle then ∇ is just d.

Generalisations
Twisted Cohomology

- For $(\lambda, \theta) \in \underline{\mathrm{Diff}}(\mathbb{R}^{0|1})$ we are given

$$\mu^*_{\lambda,\theta}(\xi) = \xi + \theta \nabla \xi,$$

- Then it follows that

$$\mu_{\lambda,\theta} : \Omega^\bullet_\nabla(M, V) \longrightarrow \Omega^\bullet_\nabla(M, V)[\theta]$$

is given by

$$\Omega^k_\nabla(M, V) \ni \omega \longmapsto (\mu^*_{\lambda,\theta})(\omega) = \lambda^k(\omega + \theta \mathrm{d}^\nabla_k \omega)$$

Generalisations

Twisted Cohomology

- With this group action we find,

$$0|1\text{-TFT}_\nabla(M,V) := \mathrm{Fun}_{\mathrm{SM}}\left(0|1\text{-B}^f(M), \underline{\pi^*(V,\nabla)}\right)$$
$$\cong \Omega^\bullet_\nabla(M,V)^{\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})}$$
$$\cong \ker\nabla.$$

- More generally we define V-valued $0|1$-field theories of degree n by

$$0|1\text{-TFT}^n_\nabla(M,V) = \Gamma(\Pi TM/\underline{\mathrm{Diff}}(\mathbb{R}^{0|1}), \pi^*(V,\nabla)\otimes(\Pi L_\rho)^{\otimes n})$$

where $\rho : \underline{\mathrm{Diff}}(\mathbb{R}^{0|1}) \to \mathbb{R}^\times$ and $L_\rho = \Pi TM \times_\rho \mathbb{R}$ is the associated line bundle over ΠTM.

Theorem

There exists bijection

$$0|1\text{-TFT}^n_\nabla(M, V) \cong \ker \mathrm{d}^\nabla_n .$$

- Theorem follows from structure of $\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})$-action.

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