IGA Lecture V: Applications to Verlinde Formulas

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Adelaide, September 9, 2011

Review

We assume G compact, simple, simply connected.

We consider q-Hamiltonian G-spaces, $\Phi\colon M \to G$:

1. $\iota(\xi_M)\omega = -\frac{1}{2}\Phi^*(\theta^L + \theta^R)\cdot\xi$,
2. $d\omega = -\Phi^*\eta$,
3. $\ker(\omega) \cap \ker(d\Phi) = 0$.

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$$(\Phi, \mathcal{E})\colon M \times \mathbb{C} \dashrightarrow \mathcal{A}^{(k)}.$$
- Then $(\Phi, \mathcal{E} \otimes \mathsf{S}^{\mathrm{op}})\colon \mathbb{C}\mathrm{l}(TM) \dashrightarrow \mathcal{A}^{(k+\mathsf{h}^\vee)}$ defines a push-forward in twisted K-homology
$$\Phi_*\colon K_0^G(M, \mathbb{C}\mathrm{l}(TM)) \to K_0^G(G, \mathcal{A}^{(k+\mathsf{h}^\vee)}).$$

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- Then $(\Phi, \mathcal{E} \otimes \mathsf{S}^{\mathrm{op}})\colon \mathbb{C}\mathrm{l}(TM) \dashrightarrow \mathcal{A}^{(k+\mathsf{h}^\vee)}$ defines a push-forward in twisted K-homology
$$\Phi_*\colon K_0^G(M, \mathbb{C}\mathrm{l}(TM)) \to K_0^G(G, \mathcal{A}^{(k+\mathsf{h}^\vee)}).$$
- The l.h.s. contains the fundamental class $[M]$. The r.h.s. is the level k fusion ring $R_k(G)$, by the FHT theorem. We define
$$\mathcal{Q}(M) := \Phi_*([M]) \in R_k(G).$$

The level k fusion ring

Notation

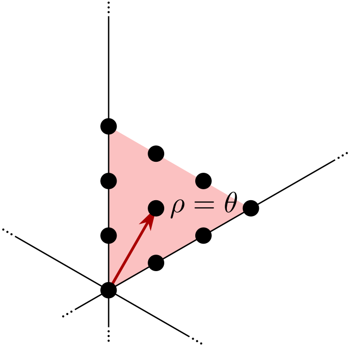
- $T \subset G$ maximal torus,
- $P_+ = P \cap \mathfrak{t}^*_+$ dominant weights,
- $\theta \in P_+$ weight of adjoint representation (highest root),
- $\cdot$ basic inner product on $\mathfrak{g} \cong \mathfrak{g}^*$: $\quad \theta \cdot \theta = 2$,
- $\rho \in P_+$ shortest weight in $P \cap \mathrm{int}(\mathfrak{t}^*_+)$,
- $\mathsf{h}^\vee = 1 + \rho \cdot \theta \quad$ dual Coxeter number

The level k fusion ring (Verlinde ring)

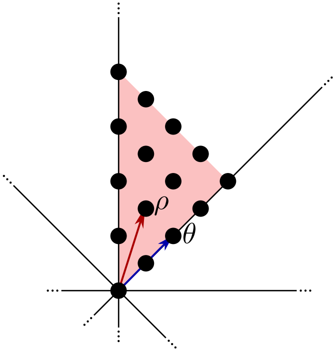
$A = \{\xi \in \mathfrak{t}_+ |\ \theta \cdot \xi \leq 1\}$ is the fundamental alcove.

Definition

The level k weights are elements of $P_k = P \cap kA$.



$G = SU(3)$
$k = 3$



$G = Spin(5)$
$k = 4$

The level k fusion ring (Verlinde ring)

For $\lambda \in P_k$ define the special element

$$t_\lambda = \exp(\frac{\lambda+\rho}{k+\mathrm{h}^\vee}) \in T.$$

Definition

The level k fusion ring (Verlinde ring) is the quotient

$$R_k(G) = R(G)/I_k(G)$$

where

$$I_k(G) = \{\chi \in R(G) |\ \chi(t_\lambda) = 0 \ \ \forall\ \lambda \in P_k\}.$$

The level k fusion ring (Verlinde ring)

Remark

$R_k(G)$ is the fusion ring of level k projective representations of the loop group LG. (But we don't need that here.)

Remark

For $G = \mathrm{SU}(r+1)$, the level k fusion ideal has generators

$$\chi_{(k+1)\varpi_1}, \cdots \chi_{(k+r)\varpi_1}$$

where $\varpi_1 \in P_+$ labels the defining representation. Similar descriptions exist for the compact symplectic groups. (Bouwknegt-Ridout, 2006)

The level k fusion ring (Verlinde ring)

Some properties of $R_k(G) = R(G)/I_k(G)$:

- $R_k(G)$ is unital ring with involution.
- $R_k(G)$ has finite $\mathbb{Z}$-basis the images τ_μ of $\chi_\mu, \mu \in P_k$. Thus

$$R_k(G) = \mathbb{Z}[P_k].$$

- $R_k(G)$ has a trace,

$$R_k(G) \to \mathbb{Z},\ \tau \mapsto \tau^G$$

  where $\tau_\mu^G = \delta_{\mu,0}$.

The level k fusion ring (Verlinde ring)

Notation

- Tensor coefficents

$$N_{\mu_1\mu_2\mu_3} = \left(\chi_{\mu_1}\chi_{\mu_2}\chi_{\mu_3}\right)^G, \quad \mu_i \in P_+$$

- Level k fusion coefficents

$$N^{(k)}_{\mu_1\mu_2\mu_3} = \left(\tau_{\mu_1}\tau_{\mu_2}\tau_{\mu_3}\right)^G, \quad \mu_i \in P_k.$$

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Then

$$N^{(k)}_{\mu_1\mu_2\mu_3} = N_{\mu_1\mu_2\mu_3}, \quad k >> 0.$$

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Example

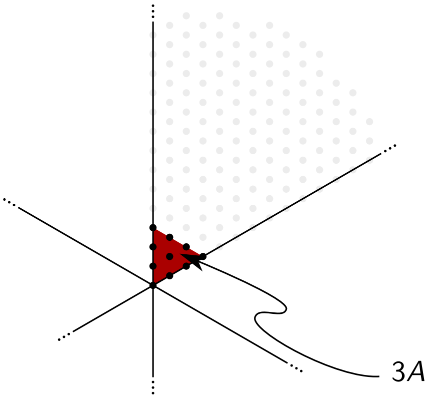
Calculation of $\tau_3 \tau_4$ in $R_5(\mathrm{SU}(2))$:

$$\chi_3 \chi_4 = \chi_7 + \chi_5 + \chi_3 + \chi_1 \Rightarrow \tau_3 \tau_4 = \tau_3 + \tau_1$$

since $\chi_7 \mapsto -\tau_5,\ \chi_5 \mapsto \tau_5$.

The level k fusion ring (Verlinde ring)

For general G the quotient map $R(G) = \mathbb{Z}[P_+] \to R_k(G) = \mathbb{Z}[P_k]$ is 'signed reflection' for a shifted Stiefel diagram.



Shifted affine Weyl action at level $k = 3$, $G = \mathrm{SU}(3)$

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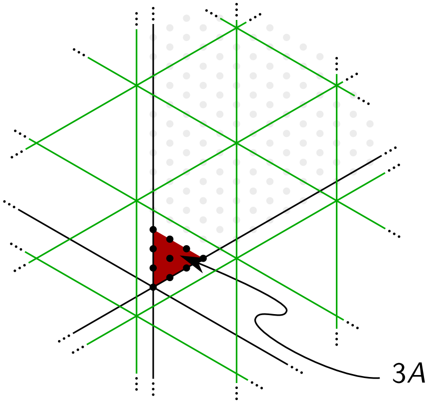

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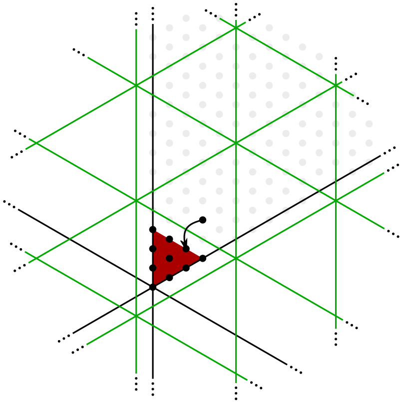

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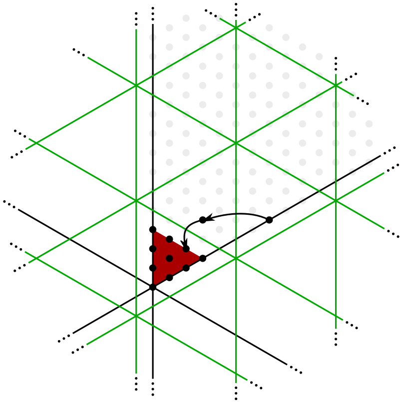

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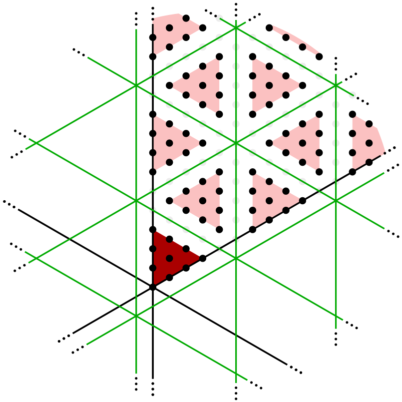

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Evaluation of characters at $t_\lambda = \exp(\frac{\lambda+\rho}{k+\mathsf{h}^\vee})$ descends to the fusion ring:

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The bases are related by the S-matrix:

$$\tau_\mu = \sum_{\nu \in P_k} S_{0,\nu}^{-1}\, S_{\mu,\nu}^* \tilde{\tau}_\nu;$$

here S is a symmetric, unitary matrix with $S_{0,\nu} > 0$.

The level k fusion ring (Verlinde ring)

$\Rightarrow$ Verlinde formula for level k fusion coefficients:

$$N^{(k)}_{\mu_1\mu_2\mu_3} = \sum_{\nu\in P_k} \frac{S_{\mu_1,\nu}S_{\mu_2,\nu}S_{\mu_3,\nu}}{S_{0,\nu}}.$$

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This is one of several formulas called 'Verlinde formulas' – this is not the difficult one.

The level k fusion ring (Verlinde ring)

Using the S-matrix, any $\tau \in R_k(G)$ may be recovered from the values $\tau(t_\lambda),\ \lambda \in P_k$:

$$\begin{aligned}\tau &= \sum_{\lambda \in P_k} \tau(t_\lambda)\tilde{\tau}_\lambda \\ &= \sum_{\lambda \in P_k} \tau(t_\lambda) S_{0,\lambda} \sum_{\mu \in P_k} S_{\lambda\mu}\tau_\mu.\end{aligned}$$

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Goal: For level k pre-quantized q-Hamiltonian G-space (M, ω, Φ), find

$$\mathcal{Q}(M)(t_\lambda) \in \mathbb{C}$$

via localization to the fixed point set.

Localization to the fixed point set

Recall $t_\lambda = \exp(\frac{\lambda+\rho}{k+\mathsf{h}^\vee})$. Let $F \subseteq M^{t_\lambda}$ a component of the fixed point set.

Since t_λ regular,

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Lemma

$\mathcal{A}^{(k+\mathsf{h}^\vee)}|_T$ *is Morita trivial; equivariantly for action of* $\langle t_\lambda \rangle \subset T$.

Localization to the fixed point set

Remark

Non-equivariantly, the Morita triviality of $\mathcal{A}^{(k+\mathsf{h}^\vee)}|_T$ *follows since the pull-back of* $\eta \in \Omega^3(G)$ *to* T *is zero. Hence*

$$H^3(G,\mathbb{Z}) \to H^3(T,\mathbb{Z})$$

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Remark

The choice of Morita trivialization of $\mathcal{A}^{(k+\mathsf{h}^\vee)}|_T$ *is not canonical in general.*

Localization to the fixed point set

Fix a $\langle t_\lambda \rangle$-equivariant Morita trivialization of $\mathcal{A}^{(k+\mathsf{h}^\vee)}|_T$.

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By composition

$$\mathbb{C}\mathrm{l}(TM|_F) \dashrightarrow \mathcal{A}^{(k+\mathsf{h}^\vee)}|_T \dashrightarrow \mathbb{C}$$

the bundle $TM|_F$ acquires a $\langle t_\lambda \rangle$-equivariant Spin_c-structure.

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$\Rightarrow$ Atiyah-Segal-Singer fixed point contributions are defined.

One proves:

Localization to the fixed point set

Let (M, ω, Φ) be a compact q-Hamiltonian G-space with a level k pre-quantization, and let $\mathcal{Q}(M) \in R_k(G)$ be its quantization.

Theorem

For all $\lambda \in P_k$,

$$\mathcal{Q}(M)(t_\lambda) = \sum_{F \subset M^{t_\lambda}} \int_F \frac{\widehat{A}(F)\,\mathrm{Ch}(\mathcal{L}_F, t_\lambda)^{1/2}}{D_{\mathbb{R}}(\nu_F, t_\lambda)}.$$

Here $\mathcal{L}_F \to F$ is the 'Spin$_c$-line bundle' for $\mathbb{C}\mathrm{l}(TM|_F) \dashrightarrow \mathbb{C}$, and ν_F is the normal bundle to F.

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One can use this formula to compute examples.

Localization to the fixed point set

Here the various characteristic forms are, in terms of curvature forms:

- $\widehat{A}(F) = \det_{\mathbb{R}}^{-1/2}(j(\frac{1}{2\pi} R_{TF})), \;\; j(z) = \frac{\sinh(z/2)}{z/2}$
- $\mathrm{Ch}(\mathcal{L}_F, t) = \mathrm{tr}_{\mathbb{C}} \left(\mu(t) \exp(\frac{1}{2\pi} R_{\mathcal{L}})\right)$
- $D_{\mathbb{R}}(\nu_F, t) = i^{\frac{1}{2}\,\mathrm{rk}(\nu_F)} \det_{\mathbb{R}}^{1/2}\Big(1 - A_F(t)^{-1} \exp(\frac{1}{2\pi} R_F)\Big).$

Here $\mu(t) \in \mathrm{U}(1)$ is the action of t on $\mathcal{L}_F$, and $A_F(t) \in \Gamma(F, \mathrm{O}(\nu_F))$ is the action of t on ν_F.

Example: conjugacy classes

Example

Let $\mathcal{C} \subset G$ be a level k pre-quantized conjugacy class. Thus $\mathcal{C} = G.\exp(\mu/k)$ where $\mu \in P_k$. Then

$$\mathcal{Q}(\mathcal{C}) = \tau_\mu,$$

the basis element of $R_k(G)$ corresponding to μ.

This is similar to the quantization of coadjoint orbits $\mathcal{O} = G.\mu \subset \mathfrak{g}^*, \mu \in P_+$.

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Both formulas are verified using the fixed point formula.

Example: The double

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The Hamiltonian analogue is the non-compact Hamiltonian G-space T^*G. Any reasonable quantization scheme for non-compact spaces gives

$$\mathcal{Q}(T^*G) = \sum_{\mu \in P_+} \chi_\mu \chi_\mu^*$$

(character for conjugation action on $L^2(G)$, defined in a completion of $R(G)$).

Example: The double

Computation of $\mathcal{Q}(D(G))$ (Sketch)

- The fixed point set of t_λ on $D(G) = G \times G$ is

$$F = D(T) = T \times T,$$

with normal bundle $\nu_F = \mathfrak{g}/\mathfrak{t} \times \mathfrak{g}/\mathfrak{t}$.

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- Since F is a torus, $\widehat{A}(F) = 1$.
- Let $J(t)$ be the Weyl denominator,
$$J(t) = \sum_{w \in W} (-1)^{l(w)} t^{w\rho} = t^\rho \prod_{\alpha > 0} (1 - t^{-\alpha}).$$
Then $\mathcal{D}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{t}, t) = J(t)$. Hence
$$D(\nu_F, t_\lambda) = |J(t_\lambda)|^2.$$

Example: The double

Computation of $\mathcal{Q}(D(G))$ (Sketch, ctd')

- The pull-back of ω to the fixed point manifold F is symplectic. One argues that $c_1(\mathcal{L}|_F)$ is a pre-quantum line bundle for the symplectic structure on F at level $2(k+\mathsf{h}^\vee)$. Hence

$$\int_F \mathrm{Ch}(\mathcal{L}, t_\lambda)^{1/2} = \int_F e^{(k+\mathsf{h}^\vee)\omega_F} = |T_{k+\mathsf{h}^\vee}|$$

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$$\mathcal{Q}(D(G))(t_\lambda) = \frac{|T_{k+\mathsf{h}^\vee}|}{|J(t_\lambda)|^2} = S_{0,\lambda}^{-2}.$$

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- Hence
$$\mathcal{Q}(D(G)) = \sum_{\lambda \in P_k} S_{0,\lambda}^{-2} \tilde{\tau}_\lambda = \sum_{\lambda \in P_k} \tau_\lambda \tau_\lambda^*.$$

Example: The sphere S^{2n}

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Recall that S^{2n} is a q-Hamiltonian SU(n)-space, pre-quantized at all levels $k \geq 0$. Fixed point set of t_λ consists of 'poles'. One computes

$$\mathcal{Q}(S^{2n}) = \tau_0 + \tau_{\varpi_1} + \cdots + \tau_{k\varpi_1}$$

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where ϖ_1 is the weight of the defining representation. This is consistent with 'quantization commutes with reduction', since S^{2n} is multiplicity free, with moment polytope is

$$\Delta(S^{2n}) = \{s\varpi_1 | \ 0 \leq s \leq 1\}.$$

Example: symplectic Verlinde formulas

Write the results for conjugacy classes and for the double in terms of the basis $\tilde{\tau}_\mu$, where $\tilde{\tau}_\mu(t_\lambda) = \delta_{\lambda,\mu}$:

$$\mathcal{Q}\big(G.\exp(\frac{1}{k}\mu)\big) = \sum_{\nu \in P_k} \frac{S^*_{\mu,\nu}}{S_{0,\nu}} \tilde{\tau}_\nu.$$

$$\mathcal{Q}(D(G)) = \sum_{\nu \in P_k} \frac{1}{S^2_{0,\nu}} \tilde{\tau}_\nu.$$

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Using $\mathcal{Q}(M_1 \times M_2) = \mathcal{Q}(M_1)\mathcal{Q}(M_2)$ this gives ...

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Let $\mu_1, \ldots, \mu_r \in P_k$, and $\mathcal{C}_j = G.\exp(\frac{1}{k}\mu_j)$. Then

$$\mathcal{Q}\Big(D(G)^h \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r\Big) = \sum_{\nu \in P_k} \frac{S^*_{\mu_1,\nu} \cdots S^*_{\mu_r,\nu}}{S_{0,\nu}^{2h+r}} \tilde{\tau}_\nu$$

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Using $\mathcal{Q}(M/\!\!/G) = \mathcal{Q}(M)^G$ and $\tilde{\tau}_\nu^G = S_{0,\nu}^2$ this gives...

Example: symplectic Verlinde formulas

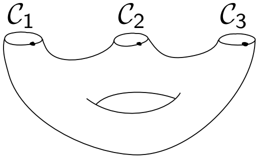
Theorem (Symplectic Verlinde formulas)

Let $\mu_1, \ldots, \mu_r \in P_k$, *and* $\mathcal{C}_j = G.\exp(\frac{1}{k}\mu_j)$. *The level* k *quantization of the moduli space*

$$\mathcal{M}(\Sigma_h^r, \mathcal{C}_1, \ldots, \mathcal{C}_r) = (D(G)^h \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r) /\!\!/ G$$

is given by

$$\mathcal{Q}\Big(\mathcal{M}(\Sigma_h^r, \mathcal{C}_1, \ldots, \mathcal{C}_r)\Big) = \sum_{\nu \in P_k} \frac{S_{\mu_1,\nu} \cdots S_{\mu_r,\nu}}{S_{0,\nu}^{2h+r-2}}$$



Verlinde formulas for non-simply connected groups

Let $Z = Z(G)$, $G' = G/Z(G)$. Then

$$D(G')^h /\!\!/ G'$$

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It has several connected components, indexed by topological types of G'-bundles.
Consider $D(G')$ as a q-Hamiltonian G-space. Then

$$D(G')^h /\!\!/ G$$

is the moduli space of flat connections on $\Sigma_h^0 \times G'$.

By D. Krepski's result, $D(G')$ is pre-quantized at level k if and only if

$$P^\vee \cdot P^\vee \subseteq \frac{1}{k}\mathbb{Z},$$

where $P^\vee = Q^*$ is the co-weight lattice. The in-equivalent pre-quantizations are indexed by $\mathrm{Hom}(Z \times Z, \mathrm{U}(1))$.

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So, what is $\mathcal{Q}(D(G'))$?

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- Let $Z \to W = N(T)/T,\ c \mapsto w_c$ be the group homomorphism defined by

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Lemma

The fixed point set of t_λ on $G' = G/Z$ is
$$(G')^{t_\lambda} = \bigcup_{c \in Z_\lambda} N(T')_c$$
where $N(T')_c \subset N(T')$ is the pre-image of $w_c \in W = N(T')/T'$.

Hence, the components of $D(G')^{t_\lambda}$ are all (left translates of) tori. Calculating the fixed point contributions requires more work. Eventually one gets:

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$$\mathcal{Q}(D(G')) = \frac{1}{|Z|^2} \sum_{c \in Z \times Z} \phi(c_1, c_2) \sum_{\lambda \in P_k^c} S_{0,\lambda}^{-2} \tilde{\tau}_\lambda$$

where $\phi(c_1, c_2)$ are phase factors depending on the pre-quantization, and $P_k^c \subset P_k$ are weights fixed by c_1, c_2.

The phase factor is explicitly

$$\phi(c_1, c_2) = \psi(c_1, c_2) e^{-2\pi i k \left((1-w_*)^{-1}\zeta_1\right)\cdot\zeta_2}$$

where $\psi \in \mathrm{Hom}(Z \times Z, \mathrm{U}(1))$ labels the pre-quantization, $\zeta_i \in A$ exponentiate to c_i, and w_* is the Coxeter transformation.

Verlinde formulas for non-simply connected groups

Following the argument for G, one finds:

Theorem (Fuchs-Schweigert formula)

$$\mathcal{Q}(D(G')^h /\!\!/ G) = \frac{1}{|Z|^{2h}} \sum_{c \in Z^{2h}} \phi(c_1, \ldots, c_{2h}) \sum_{\lambda \in P_k^c} S_{0,\lambda}^{2-2h}.$$

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Thanks!