Monodromy and orientifolds in T-duality via Courant algebroids

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The Australian National University
Canberra, Australia

Group-valued moment maps with applications to mathematics and physics
University of Adelaide, 8 September

This talk is based on

Topological T-duality for general circle bundles

arXiv:1105.0290v2

Conformal Courant algebroids and orientifold T-duality,

arXiv:1109.0875v1

and

Topological T-duality for torus bundles with monodromy,

(in preparation)

Overview

Aim of this talk is to demonstrate how the structure of **Courant algebroids** can offer some new insights into **T-duality**.

First review Courant algebroids, their relation with T-duality.

Then look at T-duality with monodromy.

Finally look at T-duality for (a very simple class of) orientifolds.

Courant algebroids

Definition

A **Courant algebroid** on a smooth manifold X consists of

- A vector bundle E,
- A bundle map $\rho : E \to TX$ called the **anchor**,
- A non-degenerate symmetric bilinear form $\langle \, , \, \rangle : E \otimes E \to \mathbb{R}$,
- An $\mathbb{R}$-bilinear operation $[\, , \,] : \Gamma(E) \otimes_{\mathbb{R}} \Gamma(E) \to \Gamma(E)$ on sections of E, the **Dorfman bracket**,

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such that for all $a, b, c \in \Gamma(E)$, $f \in \mathcal{C}^\infty(X)$

CA1 $[a, [b, c]] = [[a, b], c] + [b, [a, c]]$,

CA2 $\rho[a, b] = [\rho(a), \rho(b)]$,

CA3 $[a, fb] = \rho(a)(f)b + f[a, b]$,

CA4 $[a, b] + [b, a] = \rho^* d\langle a, b\rangle$,

CA5 $\rho(a)\langle b, c\rangle = \langle [a, b], c\rangle + \langle a, [b, c]\rangle$

Remark

The skew-symmetrisation of $[\,,\,]$ is called the **Courant bracket.**

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$\Gamma(E)$ with the Courant bracket can be made into to a Lie 2-algebra with two term complex

$$\mathcal{C}^{\infty}(X) \overset{\rho^* \circ d}{\longrightarrow} \Gamma(E)$$

Exact Courant algebroids

Definition

A Courant algebroid E is **exact** if the sequence $0 \to T^*X \xrightarrow{\rho^*} E \xrightarrow{\rho} TX \to 0$ is exact.

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Theorem (Ševera)

Isomorphism classes of exact Courant algebroids on X are in bijection with $H^3(X, \mathbb{R})$. If H is a closed 3-form on X then a representative Courant algebroid for $[H]$ is given by

- $E = TX \oplus T^*X$ *with obvious anchor and symmetric bilinear pairing*
- $[X + \xi, Y + \eta]_H = [X, Y] + \mathcal{L}_X\eta - i_Y d\xi + i_X i_Y H$

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Call $[\,,\,]_H$ the H**-twisted Dorfman bracket** on $E = TX \oplus T^*X$.

The untwisted generalised tangent bundle

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The untwisted Dorfman bracket $[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - i_Y d\xi$ makes E a Courant algebroid.

Symmetry group of E: $\mathrm{Diff}(X) \ltimes \Omega^2_{\mathrm{cl}}(X)$. A closed 2-form B acts by a B-shift:

$$e^B(X+\xi) = X + \xi + i_X B,$$

where X is a tangent vector and ξ a 1-form.

Spinors for the generalised tangent bundle

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$$S = S_+ \oplus S_-$$

where $S_+ = \bigwedge^{even} T^*X$, $S_- = \bigwedge^{odd} T^*X$.

The exterior derivative d defines a differential

$$D : \Gamma(S_\pm) \to \Gamma(S_\mp).$$

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A **graded gerbe** $\mathcal{G} = (L_{ij}, \alpha_{ij}, \theta_{ijk})$ consists of

- a $U(1)$-line bundle L_{ij} on each U_{ij},
- a $\mathbb{Z}_2$ grading for each line bundle, that is for each L_{ij} an element $\alpha_{ij} \in \mathbb{Z}_2$,
- an isomorphism $\theta_{ijk} : L_{ij} \otimes L_{jk} \to L_{ik}$ on each U_{ijk}

such that the θ_{ijk} preserve grading and satisfy the obvious associativity condition.

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such that the θ_{ijk} preserve grading and satisfy the obvious associativity condition.

The θ_{ijk} are required to respect the grading, thus $\alpha_{ij} + \alpha_{jk} = \alpha_{ik}$ and α_{ij} defines a class $\alpha \in H^1(X, \mathbb{Z}_2)$.

Graded gerbes up to stable isomorphism are classified by $H^1(X, \mathbb{Z}_2) \times H^3(X, \mathbb{Z})$.

Gerbe connections

Definition

A **connection** on $\mathcal{G}$ is a choice of unitary connection ∇_{ij} for each L_{ij} such that the θ_{ijk} are constant.

Let F_{ij} be the curvature of ∇_{ij}. The F_{ij} are closed 2-forms and

$$F_{ij} + F_{jk} + F_{ik} = 0.$$

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Define a bundle $E = E(\mathcal{G}, \nabla_{ij})$ as follows:

We set $E|_{U_i} = TX \oplus T^*X|_{U_i}$. On U_{ij} patch copies of $TX \oplus T^*X$ together using a B-shift by the closed 2-form F_{ij}.

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So a section of E is a collection $\{(X_i, \xi_i)\}$, $(X_i, \xi_i) \in \Gamma(TX \oplus T^*X, U_i)$ such that on U_{ij}

$$\begin{aligned} X_i &= X_j, \\ \xi_i &= \xi_j + i_{X_j} F_{ij}. \end{aligned}$$

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The transitions $e^{F_{ij}}$ are symmetries of $TX \oplus T^*X|_{U_i}$ as a Courant algebroid. Thus E becomes a Courant algebroid. Call E a **twisted generalised tangent bundle**.

Twisted spinor bundle

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On U_{ij} introduce the following transitions:

$$\omega_i = (-1)^{\alpha_{ij}} e^{-F_{ij}} \wedge \omega_j = (-1)^{\alpha_{ij}} (\omega_j - F_{ij} \wedge \omega_j + \frac{1}{2} F_{ij} \wedge F_{ij} \wedge \omega_j + \dots)$$

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Since the F_{ij} are **closed** we still get a differential $D : \Gamma(S_\pm) \to \Gamma(S_\mp)$.

Notice how the grading $\{\alpha_{ij}\}$ affects the *Spin*(n, n) transition functions.

Splitting the structure

Definition

A **curving** for a gerbe with connection $(\mathcal{G}, \nabla_{ij})$ is a collection of 2-forms B_i such that $B_j - B_i = F_{ij}$. There is a unique 3-form H such that $H|_{U_i} = dB_i$ called the **curvature.** $[H] \in H^3(X, \mathbb{R})$ is the image of the Dixmier-Douady class of $\mathcal{G}$ in real cohomology.

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A curving yields an explicit vector bundle isomorphism $\phi : E \simeq TX \oplus T^*X$.

Under ϕ the Courant bracket on E maps to the H-twisted Courant bracket on $TX \oplus T^*X$.

Conclude: twisting $TX \oplus T^*X$ by $\mathcal{G} \iff$ twisting $[\,,\,]$ by H.

Splitting the structure 2

A curving also yields an isomorphism of the spin bundle S:

$$S \simeq A \otimes \bigwedge^* T^* X$$

where A is the $\mathbb{Z}_2$-line bundle corresponding to $[\alpha] \in H^1(X, \mathbb{Z}_2)$.

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$$d_{\nabla,H}\omega = d_\nabla \omega + H \wedge \omega$$

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The ($\mathbb{Z}_2$-graded) cohomology groups $H^*(X, (\alpha, H))$ called the **twisted cohomology** associated to the pair $(\alpha, [H]) \in H^1(X, \mathbb{Z}_2) \times H^3(X, \mathbb{R})$.

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The ($\mathbb{Z}_2$-graded) cohomology groups $H^*(X, (\alpha, H))$ called the **twisted cohomology** associated to the pair $(\alpha, [H]) \in H^1(X, \mathbb{Z}_2) \times H^3(X, \mathbb{R})$. K-theory can be twisted by **graded** gerbes. There is a Chern character

$$Ch_{(\mathcal{G}, \nabla_{ij}, B_i)} : K^*(X, \mathcal{G}) \to H^*(X, (\alpha, H)).$$

Dimensional reduction

Let $\pi : X \to M$ be a principal circle bundle and $h \in H^3(X, \mathbb{Z})$.

Choose an invariant closed 3-form $H \in \Omega^3(X)$ representing h over $\mathbb{R}$.

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The quotient $E_{\mathrm{red}} = E/S^1$ is a vector bundle on M such that sections of E_{red} correspond to invariant sections of E.

E_{red} inherits the structure of a Courant algebroid (e.g. if a, b are invariant then so is $[a, b]_H$).

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E_{red} inherits the structure of a Courant algebroid (e.g. if a, b are invariant then so is $[a, b]_H$).

Write $E_{\mathrm{red}}(X, h)$ to indicate dependence on (X, h). Call $E_{\mathrm{red}}(X, h)$ with inherited Courant algebroid structure the **dimensional reduction** of $(E, [\,,\,]_H)$. Note that E_{red} is **not** an exact Courant algebroid.

Courant algebroids and T-duality

The link between T-duality and Courant algebroids is:

Theorem

If (X, h), $(\hat{X}, \hat{h})$ *are T-dual then the dimensional reductions* $E_{\mathrm{red}}(X, h)$, $E_{\mathrm{red}}(\hat{X}, \hat{h})$ *are isomorphic as Courant algebroids on* M.

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Definition

Let (X, h), $(\hat{X}, \hat{h})$ be principal circle bundles over M equipped with flux $h \in H^3(X, \mathbb{Z})$, $\hat{h} \in H^3(\hat{X}, \mathbb{Z})$. Then (X, h), $(\hat{X}, \hat{h})$ are **T-dual** if

- $c_1(X) = \hat{\pi}_*(\hat{h})$,
- $c_1(\hat{X}) = \pi_*(h)$,
- h and $\hat{h}$ agree when pulled back to the fibre product $C = X \times_M \hat{X}$,

where $\pi, \hat{\pi}$ are the bundle projections $\pi : X \to M$, $\hat{\pi} : \hat{X} \to M$ and $\pi_*, \hat{\pi}_*$ pushforwards in cohomology (roughly integration over the fibre).

Adding a grading to the gerbes

We can enhance this slightly by using graded gerbes: $h \in H^3(X, \mathbb{Z})$ becomes $(\alpha, h) \in H^1(X, \mathbb{Z}_2) \times H^3(X, \mathbb{Z})$.

(X, α, h), $(\hat{X}, \hat{\alpha}, \hat{h})$ are T-dual if in addition $\alpha = \hat{\alpha} \in H^1(M, \mathbb{Z}_2)$.

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Theorem

If (X, α, h), $(\hat{X}, \hat{\alpha}, \hat{h})$ are T-dual then we have isomorphisms

$$\begin{aligned} H^*(X, (\alpha, h)) &= H^{*-1}(\hat{X}, (\hat{\alpha}, \hat{h})), \\ K^*(X, (\alpha, h)) &= K^{*-1}(\hat{X}, (\hat{\alpha}, \hat{h})), \end{aligned}$$

in twisted cohomology/K-theory.

Note: for unoriented circle bundles $\alpha \neq \hat{\alpha}$ and the grading becomes necessary.

Transitive Courant algebroids

Much of the structure of (X, h) is captured by the Courant algebroid $E_{\mathrm{red}}(X, h)$ (torsion information is lost however).

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Remarkably this seems to naturally incorporate many enhancements to T-duality: monodromy, T-folds, orientifolds, heterotic T-duality, Poisson-Lie-T-duality (non-abelian T-duality).

Local description of transitive Courant algebroids

Let V be a real vector space and $\langle\,,\,\rangle$ a non-degenerate bilinear form of any signature.

Associated to V is the $\mathbb{Z}_2$-graded Clifford algebra $Cliff(V)$. Taking **graded commutators** yields a graded Lie algebra $A(V)$.

Surprising fact: $A(V)$ is actually $\mathbb{Z}$-graded!

$$A(V) = A_{-2} \oplus A_{-1} \oplus A_0 \oplus A_1 \oplus \dots$$

where $A_i = \wedge^{i+2} V$.

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On a smooth manifold M we get an associated dgla

$$\mathcal{A} = A(V) \otimes \Omega^*(M).$$

A **Maurer-Cartan element** for $\mathcal{A}$ is an element $\omega \in \mathcal{A}$ of degree 1 such that $d\omega + \frac{1}{2}[\omega, \omega] = 0$.

Local description of transitive Courant algebroids 2

A Maurer-Cartan element determines a Courant algebroid structure on $E = TM \oplus \mathcal{A}_{-1} = TM \oplus V \oplus T^*M$ through a derived bracket construction. Locally every transitive Courant algebroid has this form.

A global description just involves adding in some $O(V)$-transition functions.

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For T-duality over n-dimensional fibres let V have signature (n, n). Can write $V = t \oplus t^*$ for some rank n vector space t.

The group $O(n, n)$ acts as automorphisms of $\mathcal{A}$. This action is closely related to T-duality.

$\omega \in \Omega^3(M) \oplus (t + t^*) \otimes \Omega^2(M) \oplus \wedge^2(t + t^*) \otimes \Omega^1(M) \oplus \wedge^3(t + t^*) \otimes \Omega^0(M)$. Write as $\omega = \omega_3 + \omega_2 + \omega_1 + \omega_0$.

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We consider some special cases.

One leg on the fibre

$\omega_3 = H_3 \in \Omega^3(M)$, $\omega_2 = (F, \hat{F}) \in \Omega^2(M) \otimes (t + t^*)$, $\omega_1 = \omega_0 = 0$.

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\begin{aligned}
dF &= 0, \\
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If $A, \hat{A}$ are connections on $X, \hat{X}$ so that $dA = F$, $d\hat{A} = \hat{F}$ then

$$\begin{aligned} H &= H_3 + A \wedge \hat{F}, \\ \hat{H} &= H_3 + \hat{A} \wedge F, \end{aligned}$$

are the T-dual 3-forms on $X, \hat{X}$.

Two legs on the fibre

$\omega_0 = 0$, but now $\omega_1 \neq 0$. ω_1 is valued in $\wedge^2(t + t^*) \otimes \Omega^1(M)$.

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Impose integrality: $(F, \hat{F})$ integral and $O(n, n, \mathbb{Z})$-holonomy. Interpret as a T^{2n}-bundle $C \to M$. If the monodromy reduces to $GL(n, \mathbb{Z}) \subset O(n, n, \mathbb{Z})$ then we can write C as a fibre product $C = X \times_M \hat{X}$. Think of $X, \hat{X}$ as T-duals.

T-duality with monodromy

Courant algebroid approach suggests a T-duality for torus bundles which have both monodromy and Chern classes.

To capture this introduce the group $\mathrm{Aff}(T^n) = GL(n, \mathbb{Z}) \ltimes T^n$ of affine transformations of T^n.

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Definition

An **affine torus bundle** is a torus bundle $X \to M$ with structure group $\mathrm{Aff}(T^n)$.

Affine torus bundles have a **monodromy** representation $\rho : \pi_1(M) \to GL(n, \mathbb{Z})$ and a **twisted Chern class** $c \in H^2(M, \Lambda_\rho)$, where $\Lambda = \mathbb{Z}^n$ and Λ_ρ is the corresponding local system.

The data (ρ, c) determines X and every such pair yields and affine torus bundle. For $n \leq 3$ every torus bundle is affine.

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$$F^{3,3} \subseteq F^{2,3} \subseteq F^{1,3} \subseteq F^{0,3} = H^3(X, \mathbb{Z}).$$

Say that $h \in H^3(X, \mathbb{Z})$ is **T-dualizable** if $h \in F^{2,3}$ (h has "one leg on the fibre").

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$F^{2,3}/F^{3,3} = E_\infty^{2,3}$ is a subquotient of $E_2^{2,1} = H^2(M, (\Lambda_\rho)^*) = H^2(M, \Lambda_{\hat{\rho}})$.

This is where the dual twisted Chern class $\hat{c}$ lives, so we demand that $\hat{c}$ projects to image of h in $F^{2,3}/F^{3,3}$.

T-duality with monodromy 3

Definition

Let $\pi : X \to M$, $\hat{\pi} : \hat{X} \to M$ be affine torus bundles, monodromy $\rho, \hat{\rho}$, twisted Chern classes $c, \hat{c}$, T-dualizable fluxes $h \in H^3(X, \mathbb{Z})$, $\hat{h} \in H^3(\hat{X}, \mathbb{Z})$ and gerbe gradings $\alpha, \hat{\alpha} \in H^1(M, \mathbb{Z}_2)$.

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(X, α, h), $(\hat{X}, \hat{\alpha}, \hat{h})$ are **T-dual** if

- $\hat{\rho} = \rho^*$
- Image of $\hat{c}$ in $E_\infty^{2,1}(\pi) = F^{2,3}/F^{3,3}$ equals $h \bmod F^{3,3}$
- Similarly for c and $\hat{h}$
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- Similarly for c and $\hat{h}$
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- h and $\hat{h}$ agree on $X \times_M \hat{X}$.

Actually this last property is too weak: ((2.7) in Bunke, Rumpf, Schick).

T-duality with monodromy 4

ρ determines a flat vector bundle $V_\rho = \Lambda_\rho \otimes \mathbb{R}$. Set $w_1 = w_1(V_\rho)$, $W_3 = W_3(V_\rho)$.

Say (X, α, h) is **T-dualizable in twisted K-theory** if (β = Bockstein)

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Theorem

If (X, α, h), $(\hat{X}, \hat{\alpha}, \hat{h})$ be are T-duals of rank n, T-dualizable in twisted K-theory then we have isomorphisms

$$K^*(X, (\alpha, h)) \simeq K^{*-n}(\hat{X}, (\hat{\alpha}, \hat{h})),$$

$$H^*(X, (\alpha, h)) \simeq H^{*-n}(\hat{X}, (\hat{\alpha}, \hat{h})).$$

Example

$M = T^2$, $\pi_1(M) = \mathbb{Z}^2$ generated by x, y say.

Monodromy $\rho : \pi_1(M) \to \mathrm{SL}(2, \mathbb{Z})$ as follows:

$$\rho(x) = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}, \quad \rho(y) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

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Let $F^{2,3}(E_j) \subseteq H^3(E_j, \mathbb{Z})$ be the subgroup of T-dualisable flux. We find

$$F^{2,3}(E_j) = \begin{cases} \mathbb{Z}, & j = 0 \\ \mathbb{Z}_j, & j \neq 0 \end{cases}$$

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An integer $k \in \mathbb{Z}$ thus determines a T-dualisable flux $h_k \in H^3(E_j, \mathbb{Z})$.
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Let $K^i(j, k) = K^i(E_j, h_k)$. So

$$\begin{aligned} K^i(j,k) &= K^i(j,k+j), \\ K^i(j,k) &= K^{i-2}(k,j) = K^i(k,j) \end{aligned}$$

by B-shifts and T-duality.

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Using Leray-Serre and Atiyah-Hirzebruch we can calculate some twisted K-theory groups. An extension problem prevents calculation of the other groups.

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1	$\mathbb{Z}^6$	$\mathbb{Z}^4\oplus\mathbb{Z}_k$	$\mathbb{Z}^4\oplus\mathbb{Z}_j$	$\mathbb{Z}^4\oplus\mathbb{Z}_{\mathrm{gcd}(j,k)}$

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For $K^0(j,k)$ we make repeated use of $(j,k)\sim(j,k+j)$ and $(j,k)\sim(k,j)$. Use **Euclidean algorithm** to get $K^0(j,k)\simeq K^0(\gcd(j,k),0)=\mathbb{Z}^2\oplus\mathbb{Z}_{\gcd(j,k)}$.

Conformal Courant algebroids

Recall that $TX \oplus T^*X$ with untwisted Dorfman bracket has symmetries by closed 2-forms.

There is actually a second kind of symmetry

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where c is a non-zero constant.

This action of $\mathbb{R}^\times$ preserves the Dorfman bracket but only preserves the pairing $\langle\, ,\, \rangle$ **up to scale**.

Suggests a modification of Courant algebroid axioms that replaces $\langle\, ,\, \rangle$ by a conformal structure.

Definition

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A **conformal Courant algebroid** on a smooth manifold X consists of

- A vector bundle E,
- A line bundle L with E-connection ∇,
- A bundle map $\rho : E \to TX$ called the **anchor**,
- A non-degenerate symmetric bilinear form $\langle\, , \,\rangle : E \otimes E \to L$,
- An $\mathbb{R}$-bilinear operation $[\, , \,] : \Gamma(E) \otimes_{\mathbb{R}} \Gamma(E) \to \Gamma(E)$ on sections of E, the **Dorfman bracket**,

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such that for all $a, b, c \in \Gamma(E)$, $f \in \Gamma(L)$

CA1 $[a, [b, c]] = [[a, b], c] + [b, [a, c]]$,

CA2 $\rho[a, b] = [\rho(a), \rho(b)]$,

CA3 $[a, fb] = \rho(a)(f)b + f[a, b]$,

CA4 $[a, b] + [b, a] = \nabla\langle a, b\rangle$,

CA5 $\nabla_a\langle b, c\rangle = \langle [a, b], c\rangle + \langle a, [b, c]\rangle$

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A conformal Courant algebroid E is **exact** if the sequence
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Isomorphism classes of exact Courant algebroids on X correspond to pairs (ϵ, H) with $\epsilon \in H^1(X, \mathbb{R}^\times)$ representing a flat real line bundle L and $H \in H^3(X, L)$, modulo the equivalence $(\epsilon, H) \sim (\epsilon, cH)$ for $c \in \mathbb{R}^\times$. Given (L, ∇) and d_∇-closed 3-form H a representative Courant algebroid for $(L, \nabla), [H]$ is given by

- $E = TX \oplus (L \otimes T^*X)$ *with obvious anchor and symmetric bilinear pairing*
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Call $[\ ,\ ]_{L,H}$ the (L, H)**-twisted Dorfman bracket** on $E = TX \oplus T^*X \otimes L$.

ϵ-twisted graded gerbes

From now on take $\epsilon \in H^1(X, \mathbb{Z}_2) \subseteq H^1(X, \mathbb{R}^\times)$, so L is a flat orthogonal line bundle. Let ϵ_{ij} be a Čech cocycle representing ϵ.

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An ϵ**-twisted graded gerbe** $\mathcal{G} = (L_{ij}, \alpha_{ij}, \theta_{ijk})$ consists of

- a $\mathbb{Z}_2$-graded $U(1)$-line bundle L_{ij} on each U_{ij}
- an isomorphism $\theta_{ijk} : L_{ij}^{\epsilon_{jk}} \otimes L_{jk} \to L_{ik}$ on each U_{ijk}

such that the θ_{ijk} preserve grading and satisfies an associativity condition.

Where $L^1 = L$, $L^{-1} = L^*$.

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Graded gerbes up to stable isomorphism are classified by $H^1(X, \mathbb{Z}_2) \times H^3(X, \mathbb{Z}_\epsilon)$ where $\mathbb{Z}_\epsilon$ is the $\mathbb{Z}$-valued local system obtained from ϵ.

ϵ-twisted Gerbe connections

Definition

A **connection** on $\mathcal{G}$ is a choice of unitary connection ∇_{ij} for each L_{ij} such that the θ_{ijk} are constant.

Let F_{ij} be the curvature of ∇_{ij}. The F_{ij} are closed 2-forms and

$$\epsilon_{jk} F_{ij} + F_{jk} + F_{ik} = 0.$$

Twisted generalised tangent bundle revisited

As before obtain a bundle E which over U_i looks like $TX \oplus T^*X|_{U_i}$.

This time introduce transitions $(-1)^{\epsilon_{ij}} e^{F_{ij}} \in \mathbb{Z}_2 \ltimes \Omega^2_{\mathrm{cl}}(X)$.

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This time instead of a spin structure we consider a $\mathbb{Z}_2 \ltimes Spin(n, n)$ structure. We get a kind of $\mathbb{Z}_4$-graded spinor bundle S^t which over U_i looks like

$$S^t|_{U_i} = \bigoplus_{k \in \mathbb{Z}} L^k \otimes \bigwedge^{t+2k} T^*X$$

but globally gets twisted. Still get a differential $D : \Gamma(S^t) \to \Gamma(S^{t+1})$.

Splitting the structure

Definition

A **curving** for an ϵ-twisted gerbe with connection $(\mathcal{G}, \nabla_{ij})$ is a collection of 2-forms B_i such that $B_j - \epsilon_{ij} B_i = F_{ij}$. The locally defined 3-forms $H_i = dB_i$ satisfy $H_i = \epsilon_{ij} H_j$, so define a d_∇-closed 3-form $H \in \Omega^3(X, L)$ called the **curvature**. $[H] \in H^3(X, L)$ is the image of the Dixmier-Douady class of $\mathcal{G}$ in real cohomology.

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As before curving yields an isomorphism $\phi : E \simeq TX \oplus (L \otimes T^*X)$.

Under ϕ the Courant bracket on E maps to the (L, H)-twisted Courant bracket on $TX \oplus (L \otimes T^*X)$.

Splitting the structure 2

Similarly a curving yields an isomorphism of the spin bundle S:

$$S^i \simeq \bigoplus_{k \in \mathbb{Z}} A \otimes L^k \otimes \bigwedge^{i+2k} T^* X$$

where A is the $\mathbb{Z}_2$-line bundle A corresponding to the grading class $[\alpha] \in H^1(X, \mathbb{Z}_2)$.

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The operator D becomes the **twisted differential** $d_{\nabla,H} : \Gamma(S^i) \to \Gamma(S^{i+1})$ given by:

$$d_{\nabla,H}\omega = d_\nabla \omega + H \wedge \omega$$

where ∇ denotes the flat connection on the various $A \otimes L^k$.

Twisted cohomology

Let $H^*_\epsilon(X, (\alpha, H))$ denote the ($\mathbb{Z}_4$-graded) cohomology groups.

We conjecture that $H^*_\epsilon(X, (\alpha, H))$ is the target for a Chern character in **twisted KR-theory**.

T-duality

Let $\pi : X \to M$ be a torus bundle, $\epsilon \in H^1(M, \mathbb{Z}_2)$ and $\mathcal{G}$ an ϵ-twisted graded gerbe on X with class $(\alpha, h) \in H^1(X, \mathbb{Z}_2) \times H^3(X, \mathbb{Z}_\epsilon)$.

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Choosing a connection on X we have that

$$E_{\mathrm{red}}(X, \mathcal{G}) = TM \oplus V \oplus (L \otimes V^*) \oplus (L \otimes T^*M)$$

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Roughly speaking T-duality should interchange the inner two factors. This leads to a definition of T-duality for torus bundles with ϵ-twisted graded gerbes.

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We conjecture this extends to twisted *KR*-theory. The missing ingredients are Mayer-Vietoris and a push-forward in twisted *KR*-theory. In this case we should have an isomorphism

$$KR^i(T^2_\epsilon) = KR^{i-1}(K_\epsilon).$$

Example 2

Using a spectral sequence computation we get:

i	$KR^i(T^2_\epsilon)$	$KR^i(K_\epsilon)$
0	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}^2$
1	$\mathbb{Z}^2$	$\mathbb{Z}$
2	$\mathbb{Z}$	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$

Note: generally KR is 8-periodic, but in this example 4-periodic.

We see that $KR^i(T^2_\epsilon) = KR^{i-1}(K_\epsilon)$ is satisfied.