IGA Lecture IV: Quantization of group-valued moment maps

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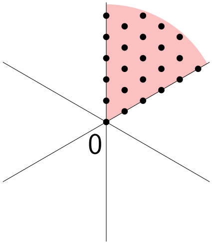
Representation ring (Notation)

The representation ring $R(G) \subset C^\infty(G)$ is the subring generated by characters χ_V of finite-dimensional G-representations V. It has basis the irreducible characters.

- G compact, connected,
- $T \subset G$ maximal torus, $\mathfrak{t} = \mathrm{Lie}(T)$,
- $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ positive Weyl chamber,
- $P \subset \mathfrak{t}^*$ (real) weight lattice,
- $P_+ = P \cap \mathfrak{t}^*_+$ dominant weights $\Rightarrow R(G) = \mathbb{Z}[P_+]$.



Quantization of Hamiltonian G-spaces

Recall axioms of Hamiltonian G-spaces, $\Phi \colon M \to \mathfrak{g}^*$:

1. $\iota(\xi_M)\omega = -\langle \mathrm{d}\Phi, \xi \rangle$,
2. $\mathrm{d}\omega = 0$,
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- Symplectic form determines a Spin_c-structure.

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Definition of quantization

- Symplectic form determines a Spin_c-structure.
- Suppose (M, ω, Φ) pre-quantizable, pick pre-quantum line bundle $L \to M$.
- Let $\partial\!\!\!/_L$ Spin_c-Dirac operator with coefficients in L. Define

$$\mathcal{Q}(M) = \mathrm{index}_G(\partial\!\!\!/_L) \in R(G).$$

Quantization of Hamiltonian G-spaces

$\mathcal{Q}(M) \in R(G)$ is independent of the choices made.

Basic Properties:

- $\mathcal{Q}(M_1 \cup M_2) = \mathcal{Q}(M_1) + \mathcal{Q}(M_2)$,
- $\mathcal{Q}(M_1 \times M_2) = \mathcal{Q}(M_1)\mathcal{Q}(M_2)$,
- $\mathcal{Q}(M^*) = \mathcal{Q}(M)^*$,
- The coadjoint orbit $G.\mu$, $\mu \in \mathfrak{t}^*_+$ is pre-quantized if and only if $\mu \in P_+$. In this case,

$$\mathcal{Q}(G.\mu) = \chi_\mu.$$

Quantization of Hamiltonian G-spaces

Let $R(G) \to \mathbb{Z}, \ \chi \mapsto \chi^G$ be the map defined by $\chi^G_\mu = \delta_{\mu,0}$.

Theorem (Quantization commutes with reduction)

Suppose M is a compact pre-quantized Hamiltonian G-space. Then

$$\mathcal{Q}(M)^G = \mathcal{Q}(M/\!\!/G).$$

This was conjectured (and proved in many cases) by Guillemin-Sternberg.

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One may take care of the singularities of $M/\!\!/G$ by partial desingularization (M-Sjamaar).

More generally, let $N(\mu),\ \mu \in P_+$ be the multiplicities given as

$$\mathcal{Q}(M) = \sum_{\mu \in P_+} N(\mu)\chi_\mu.$$

Corollary

For all $\mu \in P_+$,

$$N(\mu) = \mathcal{Q}(M/\!\!/_\mu G)$$

where

$$M/\!\!/_\mu G = \Phi^{-1}(\mathcal{O})/G = (M \times \mathcal{O}^-)/\!\!/ G.$$

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Consequences

- Let $\Delta(M) \subset \mathfrak{t}^*_+$ be the moment polytope. Then $N(\mu) = 0$ unless $\mu \in P_+ \cap \Delta(M)$.
- If M is multiplicity-free (e.g. a symplectic toric space) then $N(\mu) \in \{0, 1\}$ for all $\mu \in P_+$.

$\mathcal{Q}(M) = \mathrm{index}_G(\partial\!\!\!/)$ may also be computed by localization:

Theorem (Atiyah-Segal-Singer)

$$\mathcal{Q}(M)(g) = \sum_{F \subset M^g} \int_F \frac{\mathrm{Td}(F)\,\mathrm{Ch}(L|_F, g)}{D_{\mathbb{C}}(\nu_F, g)}$$

a sum over fixed point manifolds $F \subset M^g$.

One can also write the fixed point formula in 'Spin$_c$-form'. This will be more convenient for our discussion.

Theorem (Atiyah-Segal-Singer)

$$\mathcal{Q}(M)(g) = \sum_{F \subset M^g} \int_F \frac{\widehat{A}(F)\,\mathrm{Ch}(\mathcal{L}|_F, g)^{1/2}}{D_{\mathbb{R}}(\nu_F, g)}$$

a sum over fixed point manifolds $F \subset M^g$.

Here $\mathcal{L}$ is the 'Spin$_c$-line bundle' $\mathcal{L} = L^2 \otimes K^{-1}$, and ν_F is the normal bundle to F.

Quantization of Hamiltonian G-spaces

Here the various characteristic forms are, in terms of curvature forms:

- $\widehat{A}(F) = \det_{\mathbb{R}}^{-1/2}(j(\frac{1}{2\pi}R_{TF})),\ \ j(z) = \frac{\sinh(z/2)}{z/2}$
- $\mathrm{Ch}(\mathcal{L}|_F, t) = \mathrm{tr}_{\mathbb{C}}\left(\mu(t)\exp(\frac{1}{2\pi}R_{\mathcal{L}})\right)$
- $D_{\mathbb{R}}(\nu_F, t) = i^{\frac{1}{2}\,\mathrm{rk}(\nu_F)}\det_{\mathbb{R}}^{1/2}\Big(1 - A_F(t)^{-1}\exp(\frac{1}{2\pi}R_F)\Big).$

Here $\mu(t) \in \mathrm{U}(1)$ is the action of t on $\mathcal{L}_F$, and $A_F(t) \in \Gamma(F, \mathrm{O}(\nu_F))$ is the action of t on ν_F.

Quantization of q-Hamiltonian G-spaces ?

Recall axioms of q-Hamiltonian G-spaces, $\Phi\colon M \to G$:

1. $\iota(\xi_M)\omega = -\frac{1}{2}\Phi^*(\theta^L + \theta^R) \cdot \xi$,
2. $d\omega = -\Phi^*\eta$,
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- ω is not closed, hence 'pre-quantum line bundle' does not make sense.
- ω could be degenerate, so 'compatible almost complex structure' does not make sense. However, we constructed a 'twisted Spin_c-structure'.

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Take $\cdot$ to be the basic inner product on $\mathfrak{g}$. Then

$$\eta = \frac{1}{12}\theta^L \cdot [\theta^L, \theta^L] \in \Omega^3(G)$$

represents a generator of $H^3(G,\mathbb{Z}) \subset H^3(G,\mathbb{R})$.

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Reminder: Relative cohomology

Let $C^\bullet(X)$ denote singular cochains on X. Given $\Phi\colon X \to Y$ define

$$C^\bullet(\Phi) = C^{\bullet-1}(X) \oplus C^\bullet(Y), \quad \mathrm{d}(x, y) = (\Phi^*(y) + \mathrm{d}x, -\mathrm{d}y).$$

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Similar for Čech cohomology, de Rham cohomology, etc.

Pre-quantization

Assume G compact, 1-connected, simple.

Definition

A level k pre-quantization of a q-Hamiltonian G-space (M, ω, Φ) is a lift of $k[(\omega, \eta)] \in H^3(\Phi, \mathbb{R})$ to $H^3(\Phi, \mathbb{Z})$.

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Remark

There is an equivariant version of the definition. But since we assume $\pi_1(G) = 0$ the equivariance is automatic.

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- A pre-quantization of two q-Hamiltonian G-spaces induces a pre-quantization of their fusion product.
- The exponential of a pre-quantized Hamiltonian space inherits a pre-quantization,.
- If (M, ω, Φ) is a pre-quantized q-Hamiltonian space, and e is a regular value then $M /\!\!/ G$ is pre-quantized.

Pre-quantization: Examples

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Example

The double $D(G) = G \times G,\ \Phi(a, b) = aba^{-1}b^{-1}$ is pre-quantizable for all $k \in \mathbb{N}$, since $H_2(D(G)) = 0$.

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Example

The q-Hamiltonian SU(n)-space $M = S^{2n}$ is pre-quantized for all $k \in \mathbb{N}$, since $H_2(M) = 0$.

Pre-quantization: Examples

Recall that $P \subset \mathfrak{t}^* \cong \mathfrak{t}$ is the weight lattice, and $A \subset \mathfrak{t}_+$ the alcove.

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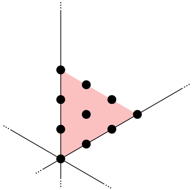
Example

A conjugacy class $\mathcal{C} = G.\exp(\xi),\ \xi \in A$ admits a level k prequantization if and only if

$$k\xi \in P_k.$$



$$G = \mathrm{SU}(3)$$
$$k = 3$$

Pre-quantization: Examples

Here is a more complicated example:

Example (D. Krepski)

Let $Z = Z(G)$, and $G' = G/Z$. Then

$$D(G') = D(G)/Z \times Z$$

is a q-Hamiltonian G-space.

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Let $Z = Z(G)$, and $G' = G/Z$. Then

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is a q-Hamiltonian G-space. Let $P^\vee$ be the co-weight lattice (dual of the root lattice). Then $D(G')$ is pre-quantizable at level k if and only if for all $\xi_1, \xi_2 \in P^\vee$,

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N.B.: $D(G')^h /\!\!/ G$ is the moduli space of flat connections on $\Sigma_h^0 \times G'$.

Pre-quantization in terms of DD bundles

Reminder: Dixmier-Douady theory

- A DD bundle $\mathcal{A} \to X$ is a $\mathbb{Z}_2$-graded bundle of C^*-algebras, with typical fiber $\mathbb{K}(H)$ (compact operators).

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- A *Morita morphism* $(\Phi, \mathcal{E})\colon \mathcal{A}_1 \dashrightarrow \mathcal{A}_2$ is a map $\Phi\colon X_1 \to X_2$ with a $\mathbb{Z}_2$-graded bundle of bimodules

$$\Phi^*\mathcal{A}_2 \circlearrowright \mathcal{E} \circlearrowleft \mathcal{A}_1,$$

modeled on $\mathbb{K}(H_2) \circlearrowright \mathbb{K}(H_1, H_2) \circlearrowleft \mathbb{K}(H_1)$.

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  modeled on $\mathbb{K}(H_2) \circlearrowright \mathbb{K}(H_1, H_2) \circlearrowleft \mathbb{K}(H_1)$.
- Up to Morita isomorphism, DD bundles over X are classified by $H^3(X, \mathbb{Z}) \times H^1(X, \mathbb{Z}_2)$.

Pre-quantization in terms of DD bundles

Relative DD bundles

In a similar way, $H^3(\Phi, \mathbb{Z}) \times H^1(\Phi, \mathbb{Z}_2)$ for $\Phi\colon X \to Y$ classifies DD bundles $\mathcal{A} \to Y$ together with Morita trivializations of the pull-back to X,

$$(\Phi, \mathcal{E})\colon X \times \mathbb{C} \dashrightarrow \mathcal{A}.$$

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For G compact, 1-connected, simple, let $\mathcal{A}^{(l)} \to G$ be trivially graded, with DD class $l \in \mathbb{Z} \cong H^3(G, \mathbb{Z})$.

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Definition

A level k pre-quantization of (M, ω, Φ) is a Morita morphism

$$(\Phi, \mathcal{E})\colon M \times \mathbb{C} \dashrightarrow \mathcal{A}^{(k)}$$

such that $\mathrm{DD}(\mathcal{E}, \mathcal{A}) \in H^3(\Phi, \mathbb{Z})$ lifts the class $[(\omega, \eta)]$. (Trivial $\mathbb{Z}_2$-gradings.)

Twisting the twisted Spin_c-structure

For Hamiltonian G-spaces, we used the pre-quantum line bundle L to twist the canonical Spin_c-structure $(p, \mathsf{S}^{\mathrm{op}})\colon \mathbb{C}\mathrm{l}(TM) \dashrightarrow \mathbb{C}$:

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We then defined $\mathcal{Q}(M) = \mathrm{index}_G(\partial\!\!\!/_L)$.

Twisting the twisted Spin_c-structure

Similarly, for a level k pre-quantized q-Hamiltonian G-space we use the pre-quantization to twist the canonical '*twisted* Spin_c-structure' $(\Phi, \mathsf{S}^{\mathrm{op}})\colon \mathbb{C}\mathrm{l}(TM) \dashrightarrow \mathcal{A}^{(\mathsf{h}^\vee)}$:

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We'll define $\mathcal{Q}(M)$ as a push-forward in twisted K-homology.

Twisted K-homology

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$$\Gamma_0(X, \mathcal{A})$$

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Definition (Donovan-Karoubi, Rosenberg)

The twisted equivariant K-homology of X with coefficients in $\mathcal{A}$ is

$$K_\bullet^G(X, \mathcal{A}) := K_G^\bullet(\Gamma_0(X, \mathcal{A})).$$

Here we are using Kasparov's definition of the K-homology of C^*-algebras:

Kasparov's definition of K-homology (Sketch)

Let A be a $\mathbb{Z}_2$-graded C^* algebra.

Definition (Atiyah, Kasparov)

A Fredholm module over A is a $\mathbb{Z}_2$-graded Hilbert space H with a $*$-representation $\pi\colon \mathrm{A} \to \mathbb{B}(H)$, together with an odd element $F \in \mathbb{B}(H)$, s.t. $\forall a \in \mathrm{A}$

1. $[\pi(a), F] \in \mathbb{K}(H)$,
2. $(F^2 + I)\pi(a) \in \mathbb{K}(H)$.

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Definition (Kasparov)

$K^0(A) =$ Fredholm modules over A, mod 'homotopy'.
$K^1(A) = K^0(A \otimes \mathbb{C}l(\mathbb{R}))$.

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Let A be a $\mathbb{Z}_2$-graded C^* algebra.

Definition (Atiyah, Kasparov)

A Fredholm module over A is a $\mathbb{Z}_2$-graded Hilbert space H with a $*$-representation $\pi\colon \mathsf{A} \to \mathbb{B}(H)$, together with an odd element $F \in \mathbb{B}(H)$, s.t. $\forall a \in \mathsf{A}$

1. $[\pi(a), F] \in \mathbb{K}(H)$,
2. $(F^2 + I)\pi(a) \in \mathbb{K}(H)$.

Definition (Kasparov)

$K^0(\mathsf{A}) =$ Fredholm modules over A, mod 'homotopy'.
$K^1(\mathsf{A}) = K^0(\mathsf{A} \otimes \mathbb{C}\mathrm{l}(\mathbb{R}))$.

We use this definition for $\mathsf{A} = \Gamma_0(X, \mathcal{A})$.

Twisted K-homology

Some remarks on $K_\bullet^G(X,\mathcal{A}) = K_G^\bullet(\Gamma_0(X,\mathcal{A}))$:

- Twisted K-homology is covariant relative to Morita morphisms $(\Phi,\mathcal{E})\colon \mathcal{A}_1 \dashrightarrow \mathcal{A}_2$ such that Φ is proper.

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- $K_0^G(\mathrm{pt}) = R(G)$, with ring structure induced by push-forward under $\mathrm{pt} \times \mathrm{pt} \to \mathrm{pt}$.
- $K_\bullet^G(X, \mathcal{A})$ is a module over $K_0^G(\mathrm{pt}) = R(G)$.

Example: elliptic operators

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Suppose D is an equivariant skew-adjoint odd elliptic differential operator acting on $V = V^+ \oplus V^- \to M$ (compact manifold).

$$H = \Gamma_{L^2}(X, V), \;\; F = \frac{D}{\sqrt{1 + D^* D}}$$

defines a K-homology class

$$[D] \in K_0^G(M).$$

The index is a push-forward under $p \colon M \to \mathrm{pt}$:

$$p_*[D] = \mathrm{index}_G(D).$$

Example: K-homology fundamental class

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Let M be a compact Riemannian G-manifold of even dimension. Then there is a fundamental class

$$[M] \in K_0^G(M, \mathbb{C}\mathrm{l}(TM)),$$

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represented by the de Rham Dirac operator on $\Gamma(M, \wedge T^*M) \cong \Gamma(M, \mathbb{C}\mathrm{l}(TM))$. Thus $\mathbb{C}\mathrm{l}(TM)$ plays the role of an 'orientation bundle' in K-theory.

Example: Freed-Hopkins-Teleman

Let G be compact, 1-connected, simple; $\mathcal{A}^{(l)} \to G$ a G-DD bundle at level $l \in \mathbb{Z} \cong H^3(G, \mathbb{Z})$.

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Theorem (Freed-Hopkins-Teleman)

For all $k \in \mathbb{Z}_{\geq 0}$, there is a canonical isomorphism of rings

$$K_0^G(G, \mathcal{A}^{(k+\mathsf{h}^\vee)}) \cong R_k(G)$$

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Additively, $R_k(G) = \mathbb{Z}[P_k]$. We'll come back to the ring structure later.

Definition of the quantization

Suppose (M, ω, Φ) is a level k pre-quantized q-Hamiltonian G-space. We had constructed

$$(\Phi, \mathcal{E} \otimes \mathsf{S}^{\mathrm{op}})\colon \mathbb{C}\mathrm{l}(TM) \dashrightarrow \mathcal{A}^{(k+\mathsf{h}^\vee)}.$$

This defines a push-forward map

$$\Phi_*\colon K_0^G(M, \mathbb{C}\mathrm{l}(TM)) \dashrightarrow K_0^G(G, \mathcal{A}^{(k+\mathsf{h}^\vee)}) \cong R_k(G).$$

Definition

The quantization of the level k pre-quantized q-Hamiltonian space (M, ω, Φ) is defined as

$$\mathcal{Q}(M) = \Phi_*([M]) \in R_k(G)$$

where $[M] \in K_0^G(M, \mathbb{C}\mathrm{l}(TM))$ is the fundamental class.

Quantization of q-Hamiltonian G-spaces

$$\mathcal{Q}(M) = \Phi_*([M]) \in R_k(G) \cong \mathbb{Z}[P_k].$$

Properties of the quantization:

- $\mathcal{Q}(M_1 \cup M_2) = \mathcal{Q}(M_1) + \mathcal{Q}(M_2)$,
- $\mathcal{Q}(M_1 \times M_2) = \mathcal{Q}(M_1)\mathcal{Q}(M_2)$,
- $\mathcal{Q}(M^*) = \mathcal{Q}(M)^*$,
- Let $\mathcal{C}$ be the conjugacy class of $\exp(\frac{1}{k}\mu)$, $\mu \in P_k$. Then

$$\mathcal{Q}(\mathcal{C}) = \tau_\mu.$$

Quantization of q-Hamiltonian G-spaces

For $\tau \in R_k(G) = \mathbb{Z}[P_k]$, let $\tau^G \in \mathbb{Z}$ be the multiplicity of τ_0.

Theorem (Quantization commutes with reduction)

Let (M, ω, Φ) be a level k prequantized q-Hamiltonian G-space. Then

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This was proved by Alekseev-M-Woodward (1999), in terms of a 'definition' of $\mathcal{Q}(M)^G$ in terms of fixed point data. Back then, we did not know how to properly define $\mathcal{Q}(M)$.

Quantization of Hamiltonian G-spaces

More generally, let $N(\mu),\ \mu \in P_k$ be the multiplicities given as

$$\mathcal{Q}(M) = \sum_{\mu \in P_k} N(\mu)\tau_\mu.$$

where $\tau_\mu \in R_k(G) = \mathbb{Z}[P_k]$ are the basis elements.

Corollary

For all $\mu \in P_k$,

$$N(\mu) = \mathcal{Q}^{(k)}(M/\!\!/_{\mathcal{C}} G)$$

where $\mathcal{C} = G.\exp(\mu/k)$, and where

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Corollary

Let $\Delta(M) \subset A$ be the moment polytope. Then $N(\mu) = 0$ unless $\mu \in P_k \cap k\Delta(M)$.