IGA Lecture I: Introduction to G-valued moment maps

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Review: Hamiltonian G-spaces

- Let G a Lie group, $\mathfrak{g} = \text{Lie}(G)$,
- $\mathfrak{g}^*$ with co-adjoint G-action denoted Ad.

Definition

A Hamiltonian G-space (M, ω, Φ) is a G-manifold M with $\omega \in \Omega^2(M)^G$ and $\Phi \in C^\infty(M, \mathfrak{g}^*)^G$ satisfying

1. $\iota(\xi_M)\omega = -\mathrm{d}\langle \Phi, \xi \rangle$,
2. $\mathrm{d}\omega = 0$,
3. $\ker(\omega) = 0$.

Examples of Hamiltonian G-spaces

Example

Coadjoint orbits $\mathcal{O} \subseteq \mathfrak{g}^*$, with Φ the inclusion.

Example

$G \times G \circlearrowright T^*G \cong G \times \mathfrak{g}^*$, with $\Phi(g, \mu) = (\operatorname{Ad}_g \mu, -\mu)$.

Example

$G \circlearrowright T^*G \cong G \times \mathfrak{g}^*$, with $\Phi(g, \mu) = (\operatorname{Ad}_g - 1)(\mu)$

Example

$\mathrm{U}(n) \circlearrowright \mathbb{C}^n$, with $\Phi \colon \mathbb{C}^n \to \mathfrak{u}(n)^*$ the map

$$\langle \Phi(z_1, \ldots, z_n), A \rangle = 2\pi i \overline{z}^\top \cdot Az, \quad A \in \mathfrak{u}(n).$$

Symplectic reduction

Meyer-Marsden-Weinstein theorem

Suppose (M, ω, Φ) is a Hamiltonian G-space, with 0 is a regular value of Φ. Then G acts locally freely on $\Phi^{-1}(0)$, and

$$M /\!\!/ G = \Phi^{-1}(0)/G$$

is a symplectic orbifold. If 0 is a singular value, then $M /\!\!/ G$ is a stratified symplectic space (Sjamaar-Lerman).

More generally, for $\mathcal{O}$ a coadjoint orbit one defines

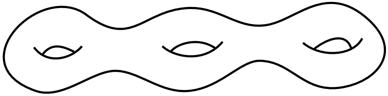
$$M /\!\!/_{\mathcal{O}} G = \Phi^{-1}(\mathcal{O})/G = (M \times \mathcal{O}^{-}) /\!\!/ G.$$

There is an extensive theory relating the geometry of $M/\!\!/G$ to the equivariant geometry of M.

- Duistermaat-Heckman (symplectic volumes)
- Kirwan, Jeffrey-Kirwan (cohomology, intersection pairings)
- Guillemin-Sternberg ('quantization commutes with reduction')
- Guillemin-Sternberg, Atiyah, Kirwan (convexity theory)
- Tolman-Weitsman
- Witten, Vergne, Paradan (non-abelian localization)
- $\cdots$

Gauge theory example (Atiyah-Bott)

- G a compact simply connected Lie group,
- $\cdot$ invariant inner product on $\mathfrak{g} = \mathrm{Lie}(G)$.

$\Sigma =$ 

$$M(\Sigma) = \frac{\{A \in \Omega^1(\Sigma, \mathfrak{g}) | \; \mathrm{d}A + \frac{1}{2}[A, A] = 0\}}{\text{gauge transformations}}$$

Motivation: Moduli Spaces of flat connections

Construction of symplectic form, after Atiyah-Bott

- $\mathcal{A}(\Sigma) = \Omega^1(\Sigma, \mathfrak{g})$ carries symplectic form $\omega(a, b) = \int_\Sigma a \cdot b$.
- $C^\infty(\Sigma, G)$ acts by gauge action,

$$g.A = \mathrm{Ad}_g(A) - \mathrm{d}g\, g^{-1},$$

- This action is Hamiltonian with moment map

$$\mathrm{curv}\colon A \mapsto \mathrm{d}A + \frac{1}{2}[A, A]$$

- Moduli space is symplectic quotient

$$M(\Sigma) = \mathrm{curv}^{-1}(0)/C^\infty(\Sigma, G).$$

Moduli Spaces of flat connections

The space $M(\Sigma) = \mathcal{A}(\Sigma) /\!\!/ C^\infty(\Sigma, G)$ is a compact singular symplectic space of dimension

$$\dim M(\Sigma) = (2h - 2) \dim G,$$

where h is the genus.

Since this reduction is infinite-dimensional, the standard techniques for symplectic quotients don't directly apply.

But $M(\Sigma)$ also has a nice finite-dimensional construction:

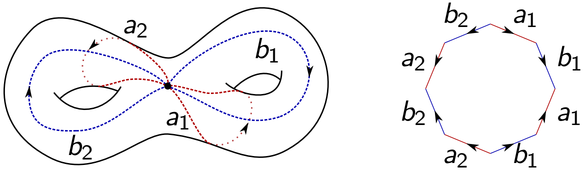
Moduli Spaces of flat connections

Holonomy description of the moduli space

$$M(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma), G)/G = \Phi^{-1}(e)/G$$

where $\Phi\colon G^{2g} \to G$ (with g the genus of Σ) is the map

$$\Phi(a_1, b_1, \ldots, a_g, b_g) = \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}.$$



We'd like to view Φ as a moment map, and $\Phi^{-1}(e)/G$ as a 'symplectic quotient' $G^{2h} /\!\!/ G$!

Group-valued moment maps

- $\theta^L = g^{-1}\,\mathrm{d}g \in \Omega^1(G, \mathfrak{g})$ left-Maurer-Cartan form
- $\theta^R = \mathrm{d}g\,g^{-1} \in \Omega^1(G, \mathfrak{g})$ right Maurer-Cartan form
- $\eta = \frac{1}{12}[\theta^L, \theta^L] \cdot \theta^L \in \Omega^3(G)$ Cartan 3-form

Definition (Alekseev-Malkin-M.)

A q-Hamiltonian G-space (M, ω, Φ) is a G-manifold M, with $\omega \in \Omega^2(M)^G$ and $\Phi \in C^\infty(M, G)^G$, satisfying

1. $\iota(\xi_M)\omega = -\frac{1}{2}\Phi^*(\theta^L + \theta^R) \cdot \xi$,
2. $\mathrm{d}\omega = -\Phi^*\eta$,
3. $\ker(\omega) \cap \ker(\mathrm{d}\Phi) = 0$.

Comparison

Hamiltonian G-space $\Phi\colon M \to \mathfrak{g}^*$

1. $\iota(\xi_M)\omega = -\mathrm{d}\langle \Phi, \xi \rangle$,
2. $\mathrm{d}\omega = 0$,
3. $\ker(\omega) = 0$.

q-Hamiltonian G-space $\Phi\colon M \to G$

1. $\iota(\xi_M)\omega = -\frac{1}{2}\Phi^*(\theta^L + \theta^R) \cdot \xi$,
2. $\mathrm{d}\omega = -\Phi^*\eta$,
3. $\ker(\omega) \cap \ker(\mathrm{d}\Phi) = 0$.

Examples: Coadjoint orbits, conjugacy classes

Example

Co-adjoint orbits $\Phi \colon \mathcal{O} \hookrightarrow \mathfrak{g}^*$ are Hamiltonian G-spaces

$$\omega(\xi_{\mathcal{O}}, \xi'_{\mathcal{O}})_\mu = \langle \mu, [\xi, \xi'] \rangle$$

Example

Conjugacy classes $\Phi \colon \mathcal{C} \hookrightarrow G$ are q-Hamiltonian G-spaces

$$\omega(\xi_{\mathcal{C}}, \xi'_{\mathcal{C}})_a = \frac{1}{2}(\mathrm{Ad}_a - \mathrm{Ad}_{a^{-1}})\xi \cdot \xi'$$

Examples; Cotangent bundle, double

Example

Cotangent bundle $T^*G \cong G \times \mathfrak{g}^*$ (with cotangent lift of $G \times G$ action) is Hamiltonian $G \times G$-space with

$$\Phi(g, \mu) = (\mathrm{Ad}_g(\mu), -\mu)$$

Example

The double $D(G) = G \times G$ is a q-Hamiltonian $G \times G$-space with action

$$(g_1, g_2).(a, b) = (g_1 a g_2^{-1}, g_2 b g_1^{-1})$$

moment map

$$\Phi(a, b) = (ab, a^{-1} b^{-1})$$

and 2-form

$$\omega = \frac{1}{2} a^* \theta^L \cdot b^* \theta^R + \frac{1}{2} a^* \theta^R \cdot b^* \theta^L.$$

Examples; Cotangent bundle, double

Example

Cotangent bundle $T^*G \cong G \times \mathfrak{g}^*$ (with cotangent lift of conjugation action) is Hamiltonian G-space with

$$\Phi(g, \mu) = \mathrm{Ad}_g(\mu) - \mu$$

Example

The double $D(G) = G \times G$ is a q-Hamiltonian G-space, with G acting by conjugation and

$$\Phi(a, b) = aba^{-1}b^{-1}.$$

Examples: Planes and spheres

Example

Even-dimensional plane $\mathbb{C}^n = \mathbb{R}^{2n}$ is Hamiltonian $\mathrm{U}(n)$-space.

Example

Even-dimensional sphere S^{2n} is a q-Hamiltonian $\mathrm{U}(n)$-space (Hurtubise-Jeffrey-Sjamaar).

Similar examples with $G = \mathrm{Sp}(n)$, and $M = \mathbb{H}^n$ resp. $\mathbb{H}P(n)$ (Eshmatov).

Basic constructions: Products

Products: If $(M_1, \omega_1, \Phi_1), (M_2, \omega_2, \Phi_2)$ are q-Hamiltonian G-spaces then so is

$$(M_1 \times M_2,\ \omega_1 + \omega_2 + \tfrac{1}{2}\Phi_1^*\theta^L \cdot \Phi_2^*\theta^R,\ \Phi_1\Phi_2).$$

Example

For instance, $D(G)^h = G^{2h}$ is a q-Hamiltonian G-space with moment map

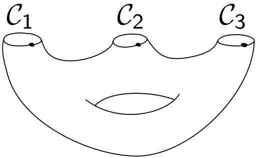
$$\Phi(a_1, b_1, \ldots, a_h, b_h) = \prod_{i=1}^{h} a_i b_i a_i^{-1} b_i^{-1}.$$

Basic constructions: Reduction

Reduction: If (M, ω, Φ) is a q-Hamiltonian G-space then the symplectic quotient

$$M /\!\!/ G := \Phi^{-1}(e)/G$$

is a symplectic manifold. with singularities



Example (and Theorem)

The symplectic quotient

$$G^{2h} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r /\!\!/ G = \mathcal{M}(\Sigma_h^r; \mathcal{C}_1, \ldots, \mathcal{C}_r)$$

is the moduli space of flat G-bundles over a surface with boundary, with boundary holonomies in prescribed conjugacy classes.

Basic constructions: Exponentials

- $\exp\colon \mathfrak{g} \to G$ exponential map,
- $\mathrm{h}\colon \Omega^\bullet(\mathfrak{g}) \to \Omega^{\bullet-1}(\mathfrak{g})$ standard homotopy operator ($\mathrm{dh} + \mathrm{hd} = \mathrm{id}$)
- $\varpi = \mathrm{h}(\exp^* \eta) \in \Omega^2(\mathfrak{g}) \Rightarrow \exp^* \eta = \mathrm{d}\varpi$.

Exponentials: If (M, ω_0, Φ_0) is a Hamiltonian G-space, such that exp regular on $\Phi_0(M) \subset \mathfrak{g}$, then

$$(M, \omega_0 - \Phi_0^* \varpi, \exp(\Phi_0))$$

is a q-Hamiltonian G-space.

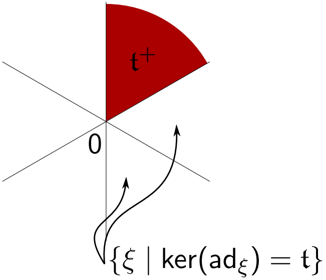
Example

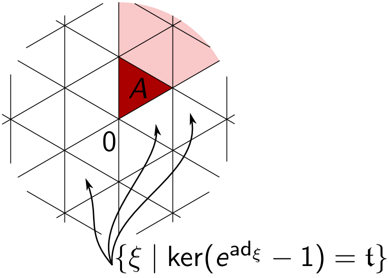
If exp regular on $\mathcal{O} \subset \mathfrak{g}^* = \mathfrak{g}$, this construction gives $\mathcal{C} = \exp(\mathcal{O})$.

Notation: Weyl chambers and Weyl alcoves

Notation

- G compact and simply connected (e.g. $G = \mathrm{SU}(n)$),
- T a maximal torus in G, $\mathfrak{t} = \mathrm{Lie}(T)$,
- $\mathfrak{t}_+ \cong \mathfrak{t}$ fundamental Weyl chamber,
- $A \subset \mathfrak{t}_+ \subset \mathfrak{t}$ fundamental Weyl alcove





Moment polytope

For every $\nu \in \mathfrak{g}^*$ there is a unique $\mu \in \mathfrak{t}^*_+$ with $\nu \in G.\mu$.

Theorem (Atiyah, Guillemin-Sternberg, Kirwan)

For a compact connected Hamiltonan G-space (M, ω, Φ), the set

$$\Delta(M) = \{\mu \in \mathfrak{t}^*_+ |\ \mu \in \Phi(M)\}$$

is a convex polytope.

For every $g \in G$ there is a unique $\xi \in A$ with $g \in G.\exp(\xi)$.

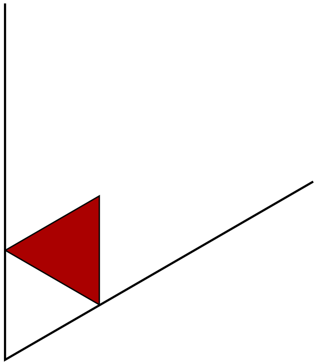
Theorem (M-Woodward)

For any connected q-Hamiltonian G-space (M, ω, Φ), the set

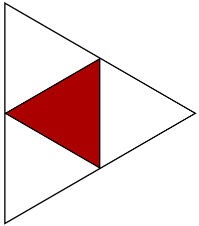
$$\Delta(M) = \{\xi \in A|\ \exp(\xi) \in \Phi(M)\}$$

is a convex polytope.

Examples of moment polytopes (due to C. Woodward)



A multiplicity-free Hamiltonian SU(3)-space



A multiplicity-free q-Hamiltonian SU(3)-space

Application to eigenvalue problems

The Hamiltonian convexity theorem gives eigenvalue inequalities for sums of Hermitian matrices with prescribed eigenvalues. (Schur-Horn conjecture; solved by Klyachko).

The q-Hamiltonian convexity theorem gives eigenvalue inequalities for products of unitary matrices with prescribed eigenvalues. (Agnihotri-Woodward).

Kirwan surjectivity

H. Cartan theorem: For $G \circlearrowright M$, $H_G(M,\mathbb{R}) = H(\Omega_G(M), \mathrm{d}_G)$ where

$$\Omega_G(M) = (S\mathfrak{g}^* \otimes \Omega(M))^G, \quad (\mathrm{d}_G\alpha)(\xi) = \mathrm{d}\alpha(\xi) - \iota(\xi_M)\alpha(\xi).$$

Theorem (Kirwan)

Let (M, ω, Φ) be a Hamiltonian G-space, with 0 regular value of Φ. Then the natural map

$$H_G(M,\mathbb{R}) \to H_G(\Phi^{-1}(0),\mathbb{R}) \cong H(M/\!\!/G,\mathbb{R})$$

is surjective.

For q-Hamiltonian spaces,

$$H_G(M,\mathbb{R}) \to H_G(\Phi^{-1}(e),\mathbb{R}) = H(M/\!\!/G)$$

is not surjective in general.

Kirwan surjecticity

Assume G simple, 1-connected.

- $\eta^i \in \Omega^{2d_i-1}(G)^G$, $i = 1, \ldots, l$ s.t. $H(G) = \wedge([\eta^1], \ldots, [\eta^l])$,
- $\eta^i_G \in \Omega_G^{2d_i-1}(G)^G$ equivariant extensions.
- Put

$$\widetilde{\Omega}_G(M) := \Omega_G(M)[u_1, \ldots, u_l], \quad \widetilde{\mathrm{d}_G} = \mathrm{d}_G + \sum \eta^i_G \frac{\partial}{\partial u_i}$$

where $\deg(u_i) = 2d_i - 2$.

Theorem (Bott-Tolman-Weitsman, Alekseev-M)

For a compact q-Hamiltonian G-space (M, ω, Φ), with e a regular value of Φ, the natural map

$$\widetilde{H}_G(M) \to H_G(\Phi^{-1}(e)) \cong H(M/\!\!/G)$$

is surjective.

Hamiltonian LG-spaces

- G compact, simply connected; metric $\cdot$ on $\mathfrak{g}$,
- $LG = \mathrm{Map}(S^1, G)$ loop group, $L\mathfrak{g} = \mathrm{Map}(S^1, \mathfrak{g})$,
- $L\mathfrak{g}^* = \Omega^1(S^1, \mathfrak{g})$ with LG acting by gauge action:

$$g.\mu = \mathrm{Ad}_g(\mu) - g^*\theta^R.$$

Definition

A Hamiltonian LG-spaces is an LG-Banach manifold $\mathcal{M}$ with a weakly symplectic form $\sigma \in \Omega^2(\mathcal{M})^{LG}$ and an equivariant map $\Psi \in C^\infty(\mathcal{M}, L\mathfrak{g}^*)$ such that

$$\sigma(\xi_{\mathcal{M}}, \cdot) + \langle \mathrm{d}\Psi, \xi \rangle = 0.$$

Note that Ψ is equivariant relative to an affine-linear action on $L\mathfrak{g}^*$.

Example (Coadjoint LG-orbits)

The coadjoint orbits $\mathcal{O} \subset L\mathfrak{g}^*$ are Hamiltonian LG-spaces, with symplectic form

$$\omega(\xi_{L\mathfrak{g}^*}(\mu), \xi'_{L\mathfrak{g}^*}(\mu)) = \langle (\mathsf{d} + \mathsf{ad}(\mu))\xi, \xi' \rangle.$$

Hamiltonian LG-spaces

Example (Moduli spaces)

- Σ surface with boundary $\partial\Sigma \cong S^1$.
- $\mathcal{G}(\Sigma) = \mathrm{Map}(\Sigma, G) \circlearrowright \mathcal{A}(\Sigma) = \Omega^1(\Sigma, \mathfrak{g})$,
- Moment map: $A \mapsto (\mathrm{curv}(A),\ i_{\partial\Sigma}^* A) \in \Omega^2(\Sigma, \mathfrak{g}) \oplus \Omega^1(\partial\Sigma, \mathfrak{g})$.
- $\mathcal{G}(\Sigma, \partial\Sigma) := \ker(\mathcal{G}(\Sigma) \to LG)$

Then

$$\mathcal{M}(\Sigma) = \mathcal{A}(\Sigma)/\!\!/\mathcal{G}(\Sigma, \partial\Sigma) = \mathrm{curv}^{-1}(0)/\mathcal{G}(\Sigma, \partial\Sigma)$$

is a Hamiltonian LG-space with $\Psi([A]) = i_{\partial\Sigma}^* A$.

Relation with Hamiltonian LG-spaces

Theorem (Alekseev-Malkin-M)

There is a 1-1 correspondence between

- *Hamiltonian LG-manifolds $(\mathcal{M}, \sigma, \Psi)$ with proper moment map $\Psi \colon \mathcal{M} \to L\mathfrak{g}^*$,*
- *compact q-Hamiltonian G-manifolds (M, ω, Φ).*

Examples of this correspondence:

- $\mathcal{M} = \mathcal{O} \subset L\mathfrak{g}^* \Rightarrow M = \mathcal{C} \subset G$,
- $\mathcal{M} = \mathcal{M}(\Sigma_h^1) \Rightarrow M = D(G)^h = G^{2h}$.

Relation with Hamiltonian LG-spaces

The '$\to$' direction of the correspondence is

$$\Phi\colon M = \mathcal{M}/L_0G \to G = L\mathfrak{g}^*/L_0G.$$

However, $\sigma \in \Omega^2(\mathcal{M})^{LG}$ is not L_0G-basic!

Fact: The pull-back of $\eta \in \Omega^3(G)$ under $L\mathfrak{g}^* \to G = L\mathfrak{g}^*/L_0G$ has a distinguished invariant primitive

$$\varpi \in \Omega^2(L\mathfrak{g}^*)^{LG}.$$

With this 2-form, $\sigma - \Psi^*\varpi$ is basic, and descends to ω.

Relation with Hamiltonian LG-spaces

Another perspective (Cabrera-Gualtieri-M 2011):

- Specify

$$L\mathfrak{g}^* = \Omega^1_{L^2}(S^1, \mathfrak{g}), \quad LG = \mathrm{Map}_{H^1}(S^1, G).$$

- Then $L\mathfrak{g}^* \to G$ is a (Banach) principal L_0G-bundle.
- Riemannian metric on $L\mathfrak{g}^*$ gives principal connection.
- By pull-back, $\mathcal{M} \to M = \mathcal{M}/L_0G$ carries a connection.
- The horizontal projection of σ is basic, and descends to ω.