Lattices in complete Kac–Moody groups

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Locally compact groups

G locally compact topological group

Examples

1. $G = (\mathbb{R}^n, +)$
2. $G = SL(2, \mathbb{R}) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \middle| a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$

Haar measure

G locally compact has left-invariant Haar measure μ
μ is unique up to scalar multiplication

Examples

1. Lebesgue measure on $G = (\mathbb{R}^n, +)$
2. $G = SL(2, \mathbb{R})$ acts on upper half-plane

$$\mathcal{U} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

by Möbius transformations $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \dfrac{az + b}{cz + d}$
Stabiliser of i is maximal compact $K = SO(2, \mathbb{R})$
Normalise μ to be compatible with this action

Lattices

G locally compact, Haar measure μ

A subgroup $\Gamma < G$ is a lattice if

- Γ is discrete
- $\mu(\Gamma \backslash G) < \infty$.

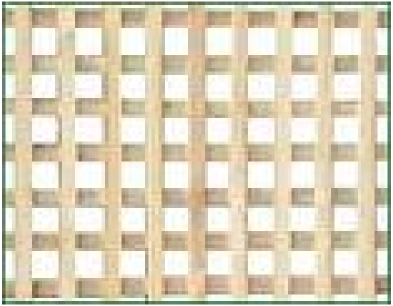
A lattice $\Gamma < G$ is

- uniform (or cocompact) if $\Gamma \backslash G$ is compact
- otherwise, nonuniform (or noncocompact).

Uniform example

Example

$\mathbb{Z}^n$ is a uniform lattice in $\mathbb{R}^n$

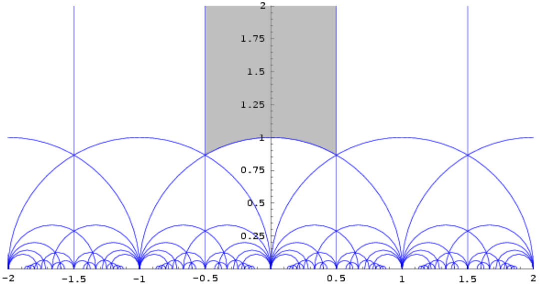


$\mu(\mathbb{Z}^n \backslash \mathbb{R}^n) = \mu(n\text{-torus}) = 1$

Nonuniform example

Example

$\Gamma = SL(2, \mathbb{Z})$ is a nonuniform lattice in $G = SL(2, \mathbb{R})$



$\mu(\Gamma \backslash G) =$ area of shaded hyperbolic triangle $= \frac{\pi}{3}$

Question

What is the set of covolumes of lattices in G?
That is, find

$$\{\mu(\Gamma\backslash G) : \Gamma \text{ a lattice in } G\} \subseteq (0, \infty)$$

Uniform/nonuniform covolumes? Lower bound?

1. $G = SL(2, K)$, K a local field
2. G the automorphism group of a locally finite tree
3. G a rank 2 complete Kac–Moody group over a finite field

Lattices in $SL(2,\mathbb{R})$

Starting point for study of covolumes:

Theorem (Siegel, 1945)

Let $G = SL(2,\mathbb{R})$. For all lattices Γ in G, $\mu(\Gamma\backslash G) \geq \frac{\pi}{21}$. This minimum is realised by a unique lattice (up to conjugacy in G), the $(2,3,7)$–triangle group, which is uniform.

Volumes of hyperbolic 3–manifolds, 3–orbifolds

$\{$covolumes of torsion-free lattices in $PSL(2, \mathbb{C})\}$

$= \{$volumes of orientable hyperbolic 3–manifolds$\}$

$\{$covolumes of lattices in $PSL(2, \mathbb{C})\}$

$= \{$volumes of orientable hyperbolic 3–orbifolds$\}$

Much studied e.g. recent work by Gabai–Meyerhoff–Milley, building on results by many others.

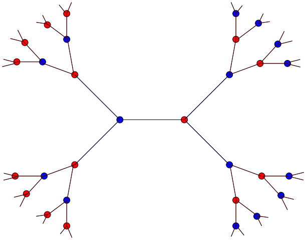
Nonarchimedean cases

Theorem (Lubotzky 1990, Lubotzky–Weigel 1999)

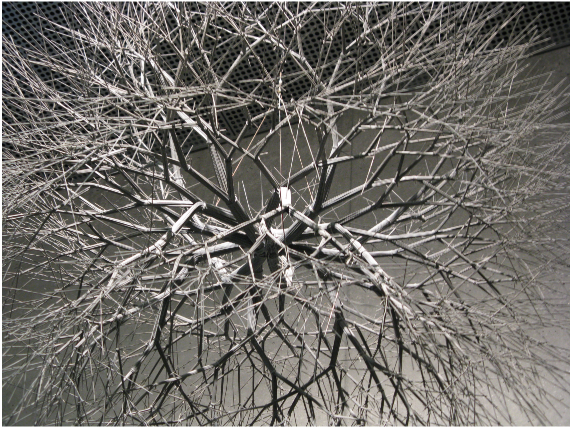
Found uniform and nonuniform lattices of minimal covolume in $G = SL(2, K)$, where K a nonarchimedean local field e.g. $K = \mathbb{Q}_p$, $K = \mathbb{F}_q((t^{-1}))$.
Lattice of minimal covolume in $G = SL(2, \mathbb{F}_q((t^{-1}))$ is the Nagao lattice $SL(2, \mathbb{F}_q[t])$, nonuniform.

Symmetric spaces and buildings

- Study real Lie groups and their lattices via action on associated symmetric space
  e.g. upper half-plane is symmetric space for $SL(2,\mathbb{R})$
- For nonarchimedean cases, use Bruhat–Tits building
  e.g. the $(q+1)$–regular tree T_{q+1} is the building for $SL(2,\mathbb{F}_q((t)))$

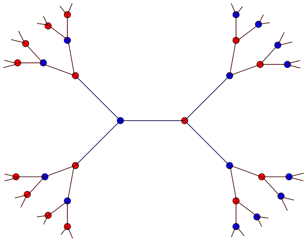


Tree lattices



Automorphism groups of trees

T locally finite tree e.g. T_3 the 3–regular tree



$G = \mathrm{Aut}(T)$, with compact-open topology, is locally compact gp.
G nondiscrete $\iff \exists \{g_n\} \subset G$ s.t. g_n fixes $\mathrm{Ball}(T, n)$, $g_n \neq 1$.

Example

$G = \mathrm{Aut}(T_3)$ nondiscrete.

Lattices in Aut(T)

T locally finite tree, $G = \mathrm{Aut}(T)$ compact-open topology

$\Gamma < G$ is discrete $\iff$ Γ acts with finite stabilisers.

Theorem (Serre)

Can normalise Haar measure μ on G so that $\forall$ discrete $\Gamma < G$

$$\mu(\Gamma \backslash G) = \sum_{v \in Vert(\Gamma \backslash T)} \frac{1}{|Stab_\Gamma(\tilde{v})|} \leq \infty$$

and Γ uniform $\iff$ $\Gamma \backslash T$ compact.

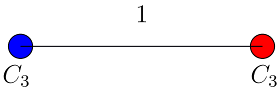
Examples of tree lattices

Example

Uniform lattice in $G = \mathrm{Aut}(T_3)$

$\Gamma = \pi_1(\text{graph of groups}) \cong C_3 * C_3$

$\mu(\Gamma \backslash G) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$



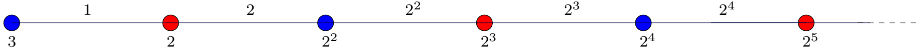
Examples of tree lattices

Example

Nonuniform lattice in $G = \mathrm{Aut}(T_3)$

$\Gamma = \pi_1(\text{graph of groups}) \cong C_3 * (\cdots)$

$\mu(\Gamma\backslash G) = \frac{1}{3} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{4}{3}$



Covolumes of tree lattices

Well-understood: see Bass–Lubotzky "Tree Lattices" (2001)

Theorem (Bass–Kulkarni, 1990)

$G = \mathrm{Aut}(T_m)$, $m \geq 3$, admits towers of uniform lattices

$$\Gamma_0 < \Gamma_1 < \Gamma_2 < \cdots < \Gamma_i < \cdots < G$$

Corollary

No positive lower bound on covolumes of (uniform) lattices in G.

Faithful amalgams

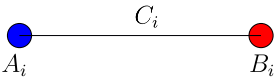
Theorem (Bass–Kulkarni, 1990)

$G = \mathrm{Aut}(T_m)$, $m \geq 3$, admits towers of uniform lattices

$$\Gamma_0 < \Gamma_1 < \Gamma_2 < \cdots < \Gamma_i < \cdots < G$$

When m composite, Γ_i are faithful (m, m)–amalgams i.e.

- $\Gamma_i = A_i *_{C_i} B_i$ with A_i, B_i, C_i finite
- $[A_i : C_i] = [B_i : C_i] = m$
- no common normal subgroup



Connection to Goldschmidt's Theorem

Theorem (Bass–Kulkarni, 1990)

$G = \mathrm{Aut}(T_m)$, $m \geq 3$, *admits towers of uniform lattices*

$$\Gamma_0 < \Gamma_1 < \Gamma_2 < \cdots < \Gamma_i < \cdots < G$$

When m composite, Γ_i are faithful (m,m)–amalgams.

Goldschmidt's Theorem (1980)

Exactly 15 faithful $(3,3)$*–amalgams:*
$C_3 * C_3,\ C_6 *_{C_2} S_3,\ S_3 *_{C_2} S_3,\ \ldots$

Conjecture (Goldschmidt–Sims): When p prime, only finitely many faithful (p,p)–amalgams.

Kac–Moody groups

In 1980s, Tits constructed functor

$$\{\text{Kac–Moody Lie algebras over } k\} \to \{\text{Kac–Moody groups over } k\}$$

Two flavours of Kac–Moody groups over $k = \mathbb{F}_q$ a finite field

- incomplete/minimal: the result of the functor
  e.g. $\Lambda = SL(n, \mathbb{F}_q[t, t^{-1}])$
- complete/topological: completion of Λ in some topology
  e.g. $G = SL(n, \mathbb{F}_q((t^{-1})))$

In general, linear representations are either not known, or do not exist (Caprace–Rémy 2009).

Incomplete Kac–Moody groups

Λ an incomplete Kac–Moody group over $\mathbb{F}_q$ e.g. $SL(n, \mathbb{F}_q[t, t^{-1}])$

- Λ is infinite but has structure similar to finite groups of Lie type
  - generated by root subgroups $U_\alpha \cong (\mathbb{F}_q, +)$
  - commutator relations
- Λ has twin Bruhat–Tits buildings $X_+ \cong X_-$ from twin BN–pairs $(B_\pm, N)$.
- Parabolic subgroups $P_\pm$ in Λ generalise $SL(n, \mathbb{F}_q[t^{\pm 1}])$ in $SL(n, \mathbb{F}_q[t, t^{-1}])$.

Complete Kac–Moody groups

An incomplete Kac–Moody group Λ over $\mathbb{F}_q$ has two completions:

$$G_+ \text{ and } G_-, \text{ with } G_+ \cong G_-$$

e.g. $\Lambda = SL(n, \mathbb{F}_q[t, t^{-1}])$ is completed to $G_\pm = SL(n, \mathbb{F}_q((t^{\pm 1})))$

The complete group $G_+ \cong G_-$ is locally compact, totally disconnected.

Work of Carbone–Garland, Caprace–Rémy, Rémy, Rémy–Ronan:

- G_+ has BN–pair $(\hat{B}_+, N)$ where $\hat{B}_+$ is completion of $B_+ < \Lambda$, and Bruhat–Tits building $\hat{X}_+ \cong X_+$, same building as for Λ.
- Kernel of G_+ action on X_+ is $Z(G_+) = Z(\Lambda)$, finite group.

and similarly for G_-.

Lattices in complete Kac–Moody groups

Let $G = G_+$ be a complete Kac–Moody group over $\mathbb{F}_q$, with building X.

Lattices in G characterised same way as lattices in $\mathrm{Aut}(T)$ i.e. subgroups $\Gamma < G$ acting on X with finite stabilisers so that

$$\mu(\Gamma\backslash G) = \sum_{v \in \mathrm{Vert}(\Gamma\backslash X)} \frac{1}{|\mathrm{Stab}_\Gamma(\tilde{v})|} < \infty$$

and Γ uniform iff Γ acts cocompactly on X.

Lattices in complete Kac–Moody groups

Let $G = G_+$ be a complete Kac–Moody group over $\mathbb{F}_q$, with building X.

Lattices in G characterised same way as lattices in $\mathrm{Aut}(T)$ i.e. subgroups $\Gamma < G$ acting on X with finite stabilisers so that

$$\mu(\Gamma\backslash G) = \sum_{v \in \mathrm{Vert}(\Gamma\backslash X)} \frac{1}{|\mathrm{Stab}_\Gamma(\tilde{v})|} < \infty$$

Very few lattices in G known.

e.g. generalising the Nagao lattice $SL(n, \mathbb{F}_q[t^{\pm}1])$ in $SL(n, \mathbb{F}_q((t^{\mp 1}))$:

Theorem (Rémy–Ronan, 2007)

$P_\pm < \Lambda$ is nonuniform lattice in $G_\mp$.

Lattices in complete Kac–Moody groups of rank 2

Theorem (Capdeboscq–T, 2010)

Let $G = G_+$ be a complete Kac–Moody group over $\mathbb{F}_q$ with symmetric generalised Cartan matrix $\begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}$, $m \geq 2$. Let $T < G$ be a fixed maximal split torus. Then

$$\min\{\, \mu(\Gamma \backslash G) : \Gamma \text{ a lattice in } G \,\} = \frac{2}{(q+1)(q-1)|T|}$$

and this min. is realised by the nonunif. lattice P_-. Moreover for $q \geq 514$,

$$\min\{\, \mu(\Gamma \backslash G) : \Gamma \text{ a uniform lattice in } G \,\} = \frac{2}{(q+1)|Z(G)|\delta}$$

with $\delta \in \{1, 2, 4\}$, and we find the unif. lattice realising this min.

Edge-transitive lattices

G has building X the $(q+1)$–regular tree.

What we did first was:

Theorem (Capdeboscq–T, 2009)

Classification of the uniform lattices in G which act transitively on the edges of X.

e.g. when $q = 2$, the only edge-transitive uniform lattice is $C_3 * C_3$.

We then showed that uniform lattices of minimal covolume are edge-transitive (most of the time), then considered nonuniform lattices.

Criterion for uniform lattices

How to recognise uniform lattices?

Theorem (The Godement Criterion)

Let G be a real semisimple Lie group. Then a lattice Γ in G is nonuniform if and only if Γ contains nontrivial unipotent elements.

This is proved using

Fact

For any locally compact group G, any uniform lattice Γ in G and any $\gamma \in \Gamma$, the set

$$\gamma^G := \{g\gamma g^{-1} \mid g \in G\}$$

is closed.

Godement Criterion in affine case

In the "affine case" $G = SL(2, \mathbb{F}_q((t^{-1})))$ considered by Lubotzky and Lubotzky–Weigel,

$$\{\text{nontrivial unipotents}\} = \{p\text{–elements}\}$$

where $q = p^a$, p prime.

Suppose lattice $\Gamma < G = SL(2, \mathbb{F}_q((t^{-1})))$ contains p–element u.

Then not hard to find $g \in G$ so that $g^n u g^{-n} \to 1_G$

e.g. $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $g = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$, $g^n u g^{-n} = \begin{pmatrix} 1 & t^{-2n} \\ 0 & 1 \end{pmatrix}$

Hence u^G not closed, so Γ nonuniform.

Conversely, if Γ nonuniform, Γ has finite subgroups of unbounded order, hence contains a p–element.

Godement Criterion for rank 2 Kac–Moody groups

Theorem (Capdeboscq–T, 2009)

Let G be a complete rank 2 Kac–Moody group over $\mathbb{F}_q$, $q = p^a$ prime, with symmetric generalised Cartan matrix. Then a lattice Γ in G is nonuniform if and only if Γ contains p–elements.

By careful analysis of action of root groups on the tree X, we show that:

- a p–element in G fixes an end of the tree X.
- the pointwise stabiliser in G of an apartment of X is torsion-free.

Using structure of end stabilisers in G, it follows that for each p–element $u \in G$, there is a $g \in G$ such that $g^n u g^{-n} \to 1_G$. Hence a lattice containing p–elements is nonuniform.