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# Singular Torus Fibrations and the Geometric origin of Mirror Symmetry

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# Outline

## I Mirror Symmetry without corrections

- Prototype
- SLAG fibration
- SYZ
- Example

## II Singular torus fibrations

- K3 example
- Local model ($\dim_{\mathbb{C}} = 2$)
- Kontsevich-Soibelman Wall crossing
- Higher dimensions
- Towards HMS

# Prototype: (Co) Tangent bundles



$B$ a smooth manifold

- $T^*B$ has a canonical symplectic form $\omega = \sum dp_i \wedge dq_i$

- Along zero section, $TB$ has almost cplex structure

$$T_{(0,b)}(TB) \cong T_bB \oplus T_bB \cong TB \otimes_{\mathbb{R}} \mathbb{C}$$

Say we have $\Lambda_b \subset T_bB$ a lattice of full rank

$\rightsquigarrow$ dual lattice $\Lambda_b^\vee \subset T_b^\vee B$.

Torus fibrations

$$\begin{tikzcd} TB/\Lambda \arrow[dr] & & T^*B/\Lambda^\vee \arrow[dl] \\ & B & \end{tikzcd}$$

# Integrability

In order for $T^\vee B/\Lambda^\vee$ to be symplectic, require

$$\Lambda^\vee_b = \mathrm{Span}\{dq_1, \dots, dq_n\} \qquad (q_1, \dots, q_n) \text{ a chart}$$

**Lemma:** Choice unique up to $GL(n, \mathbb{Z})$.

$\to$ Integral affine structure. (

**Lemma:** $TB$ and $TB/\Lambda$ are cplex manifolds

Pf: $\left(q_i, \underset{\underset{r_i}{\parallel}}{\frac{\partial}{\partial q_i}}\right)$ give a holomorphic chart.

More structure: $\Omega = (dq_1 + \sqrt{-1}\, dr_1) \wedge \dots \wedge (dq_n + \sqrt{-1}\, dr_n)$

T-duality: $(T^\vee B/\Lambda^\vee, \omega) \xleftrightarrow[\text{dual}]{} (TB/\Lambda, \Omega)$

# SLAG Fibrations



Consider $(X, J, \omega, \Omega)$, and a torus fibration
holo volume form ($n$-form)

$$\begin{array}{ccc} T^n \cong F_b & \longrightarrow & X \\ \downarrow & & \downarrow \\ b & \hookrightarrow & B \end{array}$$

- **Special:** $\Omega|_{F_b}$ = cplex valued $n$-form on $F_b$ $= e^{i\theta} \cdot \mathrm{Vol}_{F_b}$ ($\theta$ constant)
- **Lagrangian:** $\omega|_{F_b} = 0$

**Lemma:**

- $\omega$ defines a canonical isomorphism

$$T_bB \xrightarrow{\varphi} H^1(F_b, \mathbb{R}) \supset H^1(F_b, \mathbb{Z})$$

Integrate $\omega$ over annuli

- $\Omega$ defines a canonical iso

$$T_bB \xrightarrow{\psi} H^{n-1}(F_b, \mathbb{R}) \supset H^{n-1}(F_b, \mathbb{Z})$$

Integrate $\Omega$ over "n cycles"

# SYZ without correction: B-model



Define $F_b^\vee \cong \{U(1) \text{ local system on } F_b\} = H^1(F_b,\mathbb{R}) \big/ \underline{\underline{\phantom{x}}}\, H^1(F_b,\mathbb{Z})$

## Dual fibration

$$\begin{array}{ccc} F_b^\vee & \longrightarrow & X^\vee \\ & & \downarrow \\ & & B \end{array}$$

$$TX^\vee \cong TB \oplus TF_b^\vee = \underset{\substack{\cup\!\!\!\!| \\ (v,}}{TB} \oplus \underset{\alpha)}{H^1(F_b,\mathbb{R})}$$

## Complex structure: $J^\vee : TX^\vee \circlearrowleft$

$$J^\vee = \begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

$\boxed{\text{use } \omega}$

$$TX^\vee \cong H^1(F_b,\mathbb{R}) \oplus H^1(F_b,\mathbb{R})$$

## Holomorphic volume form:

$$\Omega^\vee\big((v_1,\alpha_1),\ \ldots,\ (v_n,\alpha_n)\big) = \int_{F_b} (\alpha_1 + \sqrt{-1}\,\underline{\underline{\varphi}}(v_1)) \wedge \cdots \wedge (\alpha_n + \sqrt{-1}\,\varphi(v_n))$$

# SYZ without correction: A-model 

Symplectic Form

$$\omega^\vee((v_1,\alpha_1),(v_2,\alpha_2)) = \int \alpha_1 \wedge \psi(v_2) - \alpha_2 \wedge \psi(v_1)$$

Check: $J^\vee$, $\omega^\vee$ integrable, $\Omega^\vee$ holomorphic

$\omega^\vee$ Kähler

Use affine coordinates on $B$.

↳ Legendre transform

Slogan: SYZ permutes the two affine structures

# Example: Annuli

$X_A = \{ 1 \le |z| \le e^A \} \subset \mathbb{C}^*$ $\qquad \Omega = \frac{dz}{z} = du + i d\theta$

$$\begin{array}{l} X_A \\ \Big\downarrow \log|z| \\ [0,A] \leftarrow \text{coordinate } u \end{array}$$

• Coordinates $z = e^{\underline{\underline{u+i\theta}}}$, $u \in [0, A]$.

Let $t$ be the "dual coordinate" to $\theta$

$X^\vee$ SYZ $\rightsquigarrow$ symplectic form

$\downarrow$

$[0,A]$

$$\Psi : T[0,A] \longrightarrow H^0(S^1, \mathbb{R}) = \mathbb{R}$$

$$\partial_u \longrightarrow 1$$

$$\omega^\vee(\partial_t, \partial_u) = \int_{S^1} d\theta = 1 \implies \boxed{\omega^\vee = dt \wedge du}$$

# Motivation: K3 surfaces

Recall that a K3 surface $X$ is a simply connected complex surface with $\underbrace{c_1(X)=0}_{CY}$

In codimension 1, we have an elliptic fibration

$$\begin{tikzcd} E \arrow[r] & X \arrow[d, "\pi"] \\ & \mathbb{P}^1 \end{tikzcd}$$

$\rightsquigarrow$ generically, 24 singular (nodal) fibres.


Hyperkähler $\rightsquigarrow$ $\exists$ cplex structure $J$, Kähler form $\omega_J$

holo volume form $\Omega_J$ $\ni$

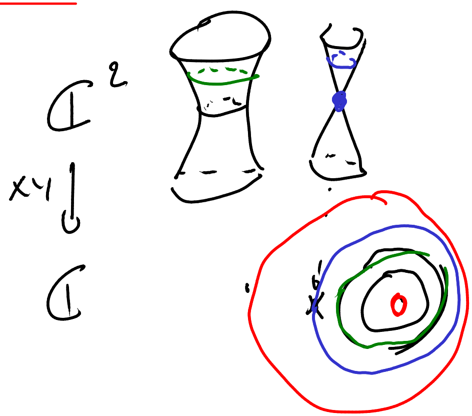
$\pi$ is SLAG fibration

# Local model

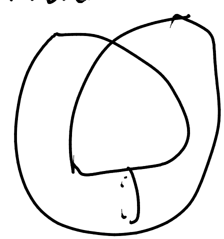
Consider $\mathbb{C}^2 \setminus \{\text{conic}\}$ with $\omega = \dfrac{dx\wedge d\bar{x} + dy\wedge d\bar{y}}{-2i}$, $\Omega = \dfrac{dx\wedge dy}{xy-1}$

Map to $\mathbb{R}\times[0,\infty)$ via $\left(|x|^2-|y|^2,\ |xy-1|\right)$. (poles at $\infty$ and $xy=1$)

Claim: Generic fibres are special Lagrangian.




- Fibres project to circles in $\mathbb{C}$ centered at $1$
- Exactly one singular fibre



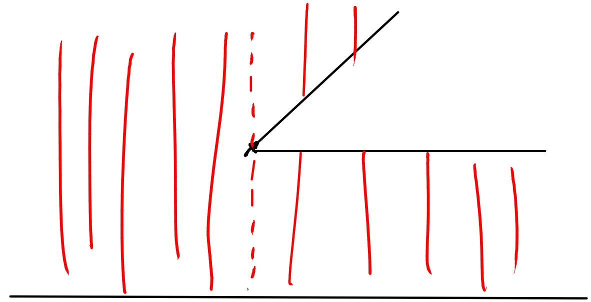
So we obtain a pair of singular affine structures on base

# Singularity of Affine Structure

- A-side: Monodromy $\begin{pmatrix}1&0\\1&1\end{pmatrix}$, remove $45°$ wedge from $\mathbb{R}\times[0,\infty)$

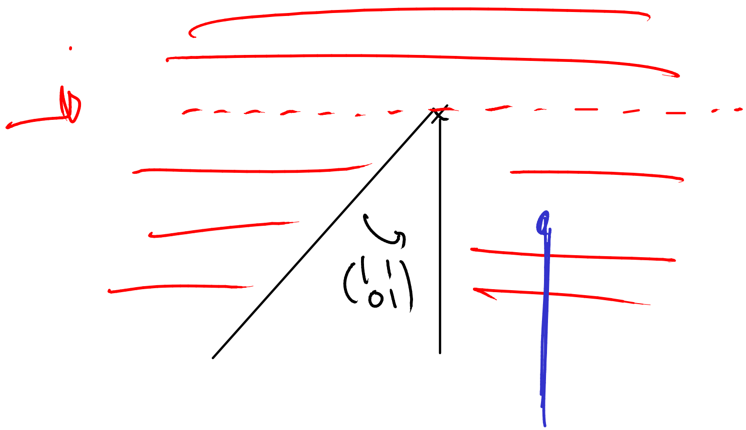
distance ~ area

Conic



- B-side: Monodromy $\begin{pmatrix}1&1\\0&1\end{pmatrix}$, remove $45°$ wedge from $\mathbb{R}^2$

- Fibres of $xy=\varepsilon$.
- One singular fibre




distance ~ conformal parameter

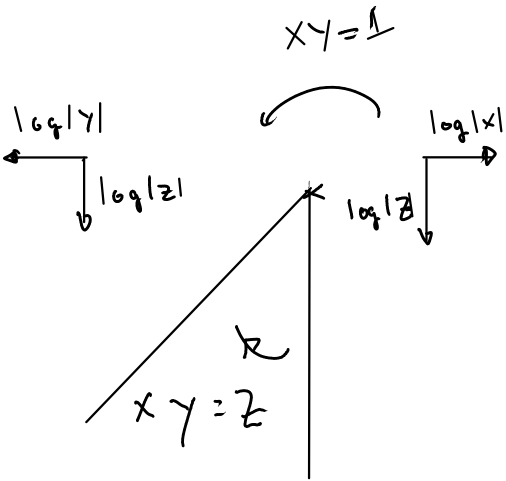
# Smoothness at Singularity

Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is integral affine, it preserves symplectic form on $T^*B/\Lambda^*$ and cplex structure on $TB/\Lambda$

Fact: • Symplectic form extends by adding singular fibre.

• Complex structure does not extend.

Solution: The naive complex gluing is incorrect!




$\rightsquigarrow$ Replace by

$$xy = 1 + z$$

# Back to K3



Start with $(B, \Lambda)$ 2-dim manifold with singular affine structure.

Goal: Construct $X$ complex manifold with torus fibration over $B$.

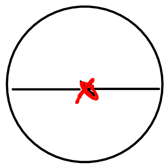
Strategy: • Construct $X_{ana}$ "rigid analytic" space then apply some version of gaga.

• Right way is scheme theoretic. See Gross-Siebert

§ Kontsevich - Soibelman 

Starting point: Fibration away from singularities, modelled after $T \to \mathbb{R}^2$. Correct the gluing.

Invariant rays emanate from each singularity



$\rightsquigarrow$ monomial $x_1 \leftrightarrow$ tangent vector $u_1$

. Say $(u_1, u_2)$ span lattice.

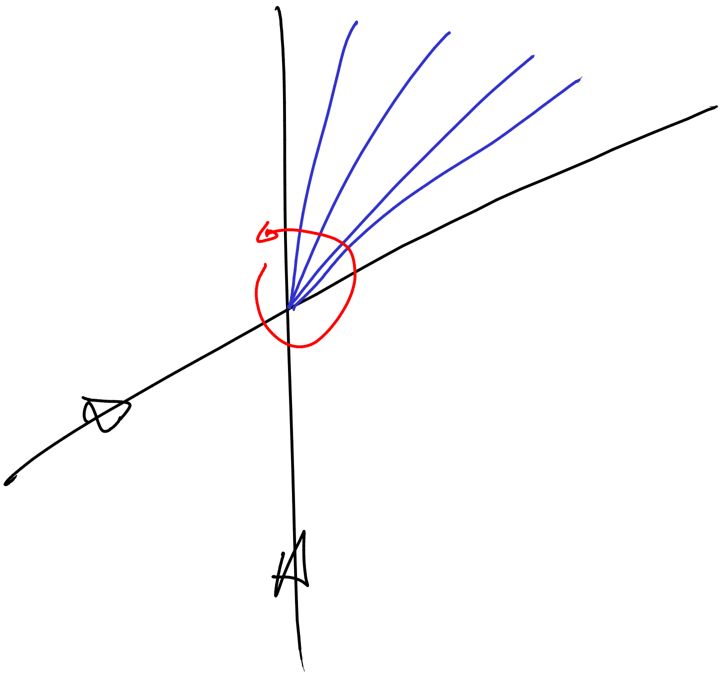
. Glue by $x_1 \longleftrightarrow x_1$

$x_2 \longleftrightarrow t(1+x_1)x_2$

Problem: When two rays meet, the gluings don't commute

# Factorisation Lemma



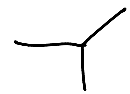


an infinite procedure
for correcting
the complex structure.

# Higher Dimensions.

<u>SYZ</u>: B should be equipped with an affine structure away from codim 2 subset $\Delta$ (discriminant locus).

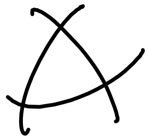
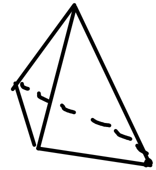
<u>e.g.</u>: In dimension 3, $\Delta$ = graph

<u>Joyce</u>:
- Even locally, there are no SLAG fibrations with discriminant locus 
- Can have thickening 

<u>Strategy</u>: Separate A & B model

# Higher Dimensions: B-model

- Say $X$ degenerates to union of toric varieties.

$E \rightsquigarrow$  Quartic $\rightsquigarrow$ 

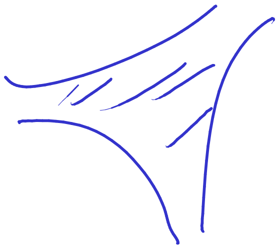
Quintic $\rightarrow \partial$(4-simplex)

- Construct singular affine structures on polytopes

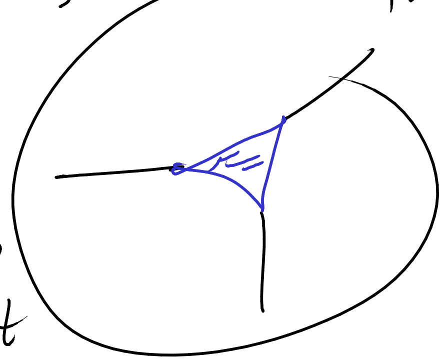
**Thm:** (Gross & Siebert) Degeneration can be reconstructed from the affine structure $+\ \varepsilon$.

# Higher Dimensions: A-model

Idea: Lagrangian fibrations ~~are~~ flexible.



deform
$\rightsquigarrow$
keeping $\omega$
constant



Glue together

Jake Solomon, Castaño Bernard ...

# § Categorical Point of View

If $X$ is CY objects of $\mathcal{F}(X)$ are $(L, \nabla)$

$\nabla$ a $U(1)$ local system.

Consider $L$ = torus.

$HF^{\bullet}((L,\nabla),(L,\nabla))$ $\qquad\qquad$ $Ext^{\bullet}(\mathcal{O}_p, \mathcal{O}_p)$

$$\begin{matrix} \| & & & & & & \| \\ HF^{\bullet}(L,L) & = & H^{\bullet}(L) & = & H^{\bullet}(T^n) & = & \Lambda^{\bullet}\mathbb{C}^n \end{matrix}$$

tori are mirror to points.

Conclusion! $X^{\vee}$ is a moduli space of objects in the Fukaya category

# Family Floer Cohomology



Fukaya: Define the mirror functor by checking that for each $L \subset X$, the groups

$$HF^*(\underline{\underline{L}}, (F_b, \nabla))$$

define a coherent sheaf

Easier approach: Identify some Lagrangians for which this makes sense

e.g.: Assume $L$ is a section of $\begin{array}{c} X \\ \downarrow \\ B \end{array}$ $\Rightarrow$

$$L \cap F_b = \{1\,pt\} \Rightarrow HF^* \text{ has rank one.}$$

$L$ is mirror to a line bundle.

Look for mirror to ample line bundle.