Homogeneous Kobayashi-Hyperbolic Manifolds

Differential Geometry Seminar 2021

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Outline

The Schwarz Lemma

We begin with a very important lemma in complex geometry, called the Schwarz lemma. Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$.

Lemma

(Schwarz Lemma). If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = 0$ then

(i) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$; and

(ii) $|f'(0)| \leq 1$.

Further, if either $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$, then f is a rotation; i.e., there exists $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} z$ for all $z \in \mathbb{D}$.

The Schwarz lemma is a key ingredient in the proof of the Riemann mapping theorem, and has many generalisations.

The Schwarz-Pick Lemma

An important generalisation of the Schwarz lemma is the *Schwarz-Pick* lemma.

Lemma

(Schwarz-Pick Lemma). If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then

1. $$\left| \frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}} \right| \leq \left| \frac{z - w}{1 - z\overline{w}} \right| \text{ for all } z, w \in \mathbb{D}$$

2. $$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2} \text{ for all } z \in \mathbb{D}.$$

with equality in each of (i) and (ii) if $f \in \mathrm{Aut}(\mathbb{D})$. If equality holds in (i) for one pair of points $z \neq w$ or if equality holds in (ii) at one point z then $f \in \mathrm{Aut}(\mathbb{D})$.

The Poincaré distance on the unit disc

The Poincaré distance ρ on the unit disc $\mathbb{D}$ is given as follows. For all $z, w \in \mathbb{D}$,

$$\rho_{\mathbb{D}}(z, w) = \tanh^{-1}\left|\frac{z - w}{1 - z\overline{w}}\right|.$$

A holomorphic mapping $f : \mathbb{D} \to \mathbb{D}$ is *distance-decreasing* under the Poincaré distance, as a consequence of the Schwarz-Pick lemma. That is,

$$\begin{aligned} \rho_{\mathbb{D}}(f(z), f(w)) &= \tanh^{-1}\left|\frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}}\right| \\ &\leq \tanh^{-1}\left|\frac{z - w}{1 - z\overline{w}}\right| \\ &= \rho_{\mathbb{D}}(z, w) \end{aligned}$$

Further, if f is biholomorphic, the distance $\rho_{\mathbb{D}}$ is invariant, that is,

$$\rho_{\mathbb{D}}(f(z), f(w)) = \rho_{\mathbb{D}}(z, w).$$

The Poincaré distance is sometimes expressed in the form

$$\rho_{\mathbb{D}}(z,w) = \frac{1}{2}\ln\frac{1+\left|\frac{z-w}{1-z\overline{w}}\right|}{1-\left|\frac{z-w}{1-z\overline{w}}\right|}.$$

The infinitesimal Poincaré metric is given by

$$\alpha_{\mathbb{D}}(z,v) = \frac{|v|}{1-|z|^2}, \text{ for } (z,v)\in T(\mathbb{D}).$$

The unit disc equipped with the Poincaré distance (or metric) is a fundamental example of a non-Euclidean geometry. That is, a geometry in which the parallel postulate of Euclidean geometry does not hold. Specifically, $(\mathbb{D},\rho_{\mathbb{D}})$ is a model of hyperbolic geometry, or *Bolyai-Lobachevskii* geometry.

Note: a mapping $\rho : D\times D \to [0,\infty)$ is a *distance* if

- $\rho(z,w)\geq 0$;
- $\rho(z,w)=\rho(w,z)$;
- $\rho(z,w)\leq d(z,v)+d(v,w)$; and
- $\rho(z,w)=0 \implies z=w$.

If the final axiom does not necessarily hold we call ρ a *pseudodistance.*

The Kobayashi pseudodistance

Let M denote a connected complex manifold.

For two points z, w in M we define a *chain* from z to w as follows: a collection of points $z = p_0, \dots, p_k = w$, points $a_1, b_1, \dots, a_k, b_k \in \mathbb{D}$ and holomorphic mappings $f_1, \dots, f_k$ from $\mathbb{D}$ into M such that $f_i(a_i) = p_{i-1}$ and $f_i(b_i) = p_i$ for all $i = 1, \dots, k$.

The Kobayashi pseudodistance K_M is then defined:

$$K_M(z, w) = \inf \left\{ \sum_{i=1}^{k} \rho_{\mathbb{D}}(a_i, b_i) : \text{ all chains in } M \right\}$$

The Kobayashi pseudodistance

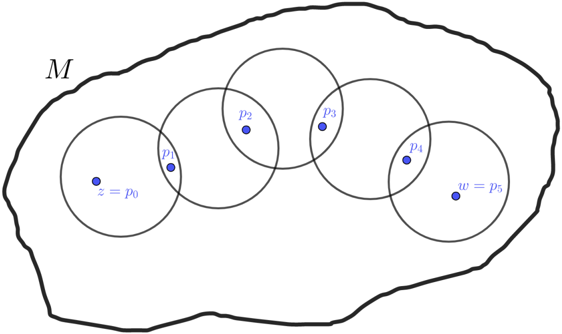

Figure: Measuring the Kobayashi pseudodistance

The Kobayashi pseudodistance is the infimum of the sum of the Poincaré distance between a_i and b_i over all such chains.

Note that K_M is a pseudodistance, and not always a true distance. It can be shown, for example, that the Kobayashi pseudodistance between two points in the complex plane is zero. That is, $K_{\mathbb{C}} \equiv 0$.

Definition

A *Kobayashi-hyperbolic* manifold is a complex manifold M for which the pseudodistance K_M is a true distance.

Some examples of manifolds that are Kobayashi-hyperbolic include:

- The unit disc $\mathbb{D}$ in the complex plane,
- The unit ball B^n in $\mathbb{C}^n$,
    - $B^n := \left\{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 < 1\right\}$
- Any bounded domain in $\mathbb{C}^n$,
- For example, the domain $\Omega := \left\{(z, w) \in \mathbb{C}^2 : |z| < 1, |w| < \frac{1}{1-|z|}\right\}$.

Some examples of manifolds that are *not* Kobayashi-hyperbolic include:

- $\mathbb{C}$ and $\mathbb{C}^*$,
- $\mathbb{C}^n$,
- $\mathbb{CP}^n$,
- Any Oka manifold.

Importance

The Kobayashi pseudodistance is a pseudodistance on an arbitrary complex manifold that in some sense captures the natural geometry of the Poincaré distance on the unit disc.

The Kobayashi pseudodistance agrees with the Poincaré distance on the unit disc.

More importantly, the Kobayashi pseudodistance shares with the Poincaré distance the distance-decreasing property for holomorphic maps, that is, for $f : M_1 \to M_2$ we have

$$K_{M_2}(f(z), f(w)) \leq K_{M_1}(z, w)$$

for all $z, w \in M_1$. Similarly, we have equality where f is a biholomorphism. That is,

$$K_{M_2}(f(z), f(w)) = K_{M_1}(z, w)$$

for all $z, w \in M_1$.

Aut(M) is a Lie group

Consider a biholomorphic automorphism $f : M \to M$. Then, of course,

$$K_M(f(z), f(w)) = K_M(z, w)$$

for all $z, w \in M$, and we say that K_M is Aut(M)-invariant. This fact allows us to prove a remarkable fact about the automorphism group of a Kobayashi-hyperbolic manifold, which we now turn to.

Theorem

If M is Kobayashi-hyperbolic, its automorphism group Aut(M) *is a real Lie group in the compact-open topology.*

Proof (sketch):

- It was shown by Kaup in 1967 that the action of Aut(M) on M is *proper* if M admits a continuous Aut(M)-invariant distance function.
- The fact that the action of Aut(M) on M is proper implies that Aut(M) is locally compact.
- It was shown by Bochner and Montgomery in 1946 that Aut(M) is a Lie group whenever it is locally compact.

Classification of hyperbolic manifolds

We are interested in obtaining a complete classification of Kobayashi-hyperbolic manifolds M with high-dimensional automorphism group.

Let $n := \dim_{\mathbb{C}} M$.

Let $d(M) := \dim \mathrm{Aut}(M)$, i.e., $d(M)$ is the real dimension of the automorphism group of M.

In 1967, Kaup established an upper bound on $d(M)$, given by $d(M) \leq n^2 + 2n$, with equality if and only if M is biholomorphic to B^n.

As it turns out, there is a lacuna located immediately below the maximum possible dimension. By a theorem of Isaev and Krantz from 2001, there exist no Kobayashi-hyperbolic manifolds M with $n^2 + 3 \leq d(M) < n^2 + 2n$.

In the early 2000s, Isaev completed an explicit classification of Kobayashi-hyperbolic manifolds for $d(M)$ in the range $n^2 - 1 \leq d(M) \leq n^2 + 2$.

At the value $d(M) = n^2 - 2$ the classification becomes infeasible. To see this, consider *Reinhardt domains* in $\mathbb{C}^n$, that is, domains invariant under the rotations

$$z_j \mapsto e^{i\varphi_j} z_j \text{ where } \varphi_j \in \mathbb{R} \text{ for } j = 1, \ldots, n.$$

Most Reinhardt domains have no automorphisms other than these rotations, and hence have an n-dimensional automorphism group. In particular, if D is a typical Reinhardt domain in $\mathbb{C}^2$, then $d(D) = 2 = n^2 - 2$. Such domains have uncountably many isomorphism classes, and so cannot be explicitly described.

Homogeneity

We can continue the classification if we introduce the assumption of *homogeneity*. A manifold M is *homogeneous* if the action of $\mathrm{Aut}(M)$ on M is *transitive.*

- The group $\mathrm{Aut}(M)$ acts transitively on M if for every $p, q \in M$ there exists a $f \in \mathrm{Aut}(M)$ such that $fp = q$.

Placing this extra structure on Kobayashi-hyperbolic manifolds allows us to exploit some very powerful results.

One more definition...

A *Siegel domain of the second kind* is an unbounded domain in $\mathbb{C}^n$ of the form

$$S(\Omega, H) := \left\{(z, w) \in \mathbb{C}^k \times \mathbb{C}^{n-k} : \operatorname{Im} z - H(w, w) \in \Omega\right\}$$

for some $1 \leq k \leq n$, some open convex cone $\Omega \subset \mathbb{R}^k$, and some Ω-Hermitian form $H = (H_1, ..., H_k)$ on $\mathbb{C}^{n-k}$.

- An open subset $\Omega \subset \mathbb{R}^k$ is an *open convex cone* if it is closed with respect to taking linear combinations of its elements with positive coefficients.
- A Hermitian form H is called Ω-Hermitian if $H(w, w) \in \bar{\Omega} \setminus \{0\}$ for all non-zero $w \in \mathbb{C}^{n-k}$.

A Siegel domain of the second kind is a multidimensional analogue of the upper half-plane in $\mathbb{C}$. Every Siegel domain of the second kind has a bounded realisation, hence is Kobayashi-hyperbolic.

Two structure theorems

It was proved by Vinberg, Gindikin and Pyatetskii-Shapiro in 1963 that every homogeneous bounded domain in $\mathbb{C}^n$ is biholomorphic to an affinely homogeneous Siegel domain of the second kind.

This result was extended by Nakajima in 1985, where it was proved that every homogeneous Kobayashi-hyperbolic manifold is biholomorphic to an affinely homogeneous Siegel domain of the second kind.

The Automorphism group of a Siegel domain

Let $G(\Omega)$ denote the automorphism group of the cone Ω.

Any holomorphic affine automorphism of $S(\Omega, H)$ has the form

$$(z, w) \mapsto (Az + a + 2iH(b, Bw) + iH(b, b), Bw + b),$$

with $a \in \mathbb{R}^k$, $b \in \mathbb{C}^{n-k}$, $A \in G(\Omega)$, $B \in \mathrm{GL}_{n-k}(\mathbb{C})$, where

$$AH(w, w') = H(Bw, Bw')$$

for all $w, w' \in \mathbb{C}^{n-k}$.

A Siegel domain is *affinely homogeneous* if the above group of holomorphic affine automorphisms acts on it transitively. We have an explicit description of the Lie algebra of the above automorphism group, denoted $\mathfrak{g}(S(\Omega, H))$, which we consider now.

The algebra $\mathfrak{g} = \mathfrak{g}(S(\Omega, H))$ admits a grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1,$$

with $\mathfrak{g}_\nu$ being the eigenspace with eigenvalue ν of $\operatorname{ad}\partial$, where

$$\partial := z \cdot \frac{\partial}{\partial z} + \frac{1}{2} w \cdot \frac{\partial}{\partial w}.$$

Here

$$\mathfrak{g}_{-1} = \left\{ a \cdot \frac{\partial}{\partial z} : a \in \mathbb{R}^k \right\}, \qquad \dim \mathfrak{g}_{-1} = k,$$

$$\mathfrak{g}_{-1/2} = \left\{ 2iH(b, w) \cdot \frac{\partial}{\partial z} + b \cdot \frac{\partial}{\partial w} : b \in \mathbb{C}^{n-k} \right\}, \quad \dim \mathfrak{g}_{-1/2} = 2(n-k),$$

and $\mathfrak{g}_0$ consists of all vector fields of the form

$$(Az) \cdot \frac{\partial}{\partial z} + (Bw) \cdot \frac{\partial}{\partial w},$$

with $A \in \mathfrak{g}(\Omega)$, $B \in \mathfrak{gl}_{n-k}(\mathbb{C})$ and

$$AH(w, w') = H(Bw, w') + H(w, Bw')$$

for all $w, w' \in \mathbb{C}^{n-k}$.

Work involved

Explicit descriptions of the components $\mathfrak{g}_{1/2}$ and $\mathfrak{g}_1$ are also known, but we omit them.

We provide a brief illustration of the work involved in determining the contributions made to the classification. We begin with the given automorphism group dimension. Take as an example $d(M) = n^2 - 3$. Recall that the definition of a Siegel domain is given by

$$S(\Omega, H) := \left\{ (z, w) \in \mathbb{C}^k \times \mathbb{C}^{n-k} : \operatorname{Im} z - H(w, w) \in \Omega \right\}.$$

Dimensional considerations allow us to rule out a large number of possibilities. In this case, the cases we need to consider are the following:

- $k = 2, n \geq 4$,
- $k = 3, n = 4$,
- $k = 3, n = 5$,
- $k = 4, n = 4$.

Let us take as an example $k = 3, n = 4$.

Homogeneous open convex cones

$k = 2$: $\Omega_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 > 0\}$, where the algebra $\mathfrak{g}(\Omega_1)$ consists of all diagonal matrices, hence $\dim \mathfrak{g}(\Omega_1) = 2$,

$k = 3$:
- (i) $\Omega_2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0,\ x_2 > 0,\ x_3 > 0\}$, where the algebra $\mathfrak{g}(\Omega_2)$ consists of all diagonal matrices, hence $\dim \mathfrak{g}(\Omega_2) = 3$,
- (ii) $\Omega_3 := \Lambda_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 - x_2^2 - x_3^2 > 0,\ x_1 > 0\}$, where one has $\mathfrak{g}(\Omega_3) = \mathfrak{c}(\mathfrak{gl}_3(\mathbb{R})) \oplus \mathfrak{so}_{1,2}$, hence $\dim \mathfrak{g}(\Omega_3) = 4$; here for any Lie algebra $\mathfrak{h}$ we denote by $\mathfrak{c}(\mathfrak{h})$ its centre,

$k = 4$:
- (i) $\Omega_4 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 > 0,\ x_2 > 0,\ x_3 > 0,\ x_4 > 0\}$, where the algebra $\mathfrak{g}(\Omega_4)$ consists of all diagonal matrices, hence we have $\dim \mathfrak{g}(\Omega_4) = 4$,
- (ii) $\Omega_5 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 - x_2^2 - x_3^2 > 0,\ x_1 > 0,\ x_4 > 0\}$, where the algebra $\mathfrak{g}(\Omega_5) = (\mathfrak{c}(\mathfrak{gl}_3(\mathbb{R})) \oplus \mathfrak{so}_{1,2}) \oplus \mathbb{R}$ consists of block-diagonal matrices with blocks of sizes 3×3 and 1×1 corresponding to the two summands, hence $\dim \mathfrak{g}(\Omega_5) = 5$,
- (iii) $\Omega_6 := \Lambda_4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 - x_2^2 - x_3^2 - x_4^2 > 0,\ x_1 > 0\}$, where $\mathfrak{g}(\Omega_6) = \mathfrak{c}(\mathfrak{gl}_4(\mathbb{R})) \oplus \mathfrak{so}_{1,3}$, hence $\dim \mathfrak{g}(\Omega_6) = 7$.

Take $k = 3, n = 4$. Then $S(\Omega, H)$ is given by

$$S(\Omega, H) = \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C} : \operatorname{Im} z - H(w, w) \in \Omega\right\}.$$

So we need to consider every Siegel domain corresponding to each cone in $\mathbb{R}^3$. These are the following two:

$$D = \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C} : \operatorname{Im} z - H(w, w) \in \Omega_2\right\}$$

and

$$D' = \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C} : \operatorname{Im} z - H(w, w) \in \Omega_3\right\}.$$

Typically, we proceed by computing $\mathfrak{g}_{1/2}$ or $\mathfrak{g}_1$ for the domain (and hence the dimension of the Lie algebra) and comparing with the automorphism group dimension (in this case $d(M) = n^2 - 3$).

As the automorphism group dimension progresses downwards, we need to consider homogeneous open convex cones of higher and higher dimension. At a certain point, it was necessary to consider five-dimensional cones.

In particular, the following two:

$$\Omega_7 := \big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 > 0, \;\; x_1 x_2 - x_4^2 > 0, \\ x_1 x_2 x_3 - x_3 x_4^2 - x_2 x_5^2 > 0\big\},$$

and

$$\Omega_8 := \big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 > 0, \;\; x_1 x_2 - x_4^2 > 0, \;\; x_1 x_3 - x_5^2 > 0\big\}\,.$$

These are referred to as the *Vinberg cone* and *dual Vinberg cone* in the literature.

We provide a table of the classification so far. In doing so, we use the following notation:

Let B^1 denote the open unit disc $\mathbb{D}$ in $\mathbb{C}$.

Let B^2 denote the open unit ball in $\mathbb{C}^2$, etc.

Note further that

$$T_3 := \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : (\mathrm{Im}\, z_1)^2 - (\mathrm{Im}\, z_2)^2 - (\mathrm{Im}\, z_3)^2 > 0, \mathrm{Im}\, z_1 > 0\right\},$$

$$\begin{aligned} T_4 : = \big\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : (\mathrm{Im}\, z_1)^2 - (\mathrm{Im}\, z_2)^2 - (\mathrm{Im}\, z_3)^2 \\ - (\mathrm{Im}\, z_4^2) > 0, \mathrm{Im}\, z_1 > 0\big\}, \end{aligned}$$

and so on.

The contributions on the following slide are due to Isaev.

$d(M)$	Contribution
n^2-2	$B^1 \times B^1 \times B^2$
	$B^2 \times B^3$
n^2-3	$T_3 \times B^1$
n^2-4	$B^1 \times B^1 \times B^1 \times B^1$
	$B^1 \times B^1 \times B^3$
	T_5
	$B^2 \times B^4$
n^2-5	NONE
n^2-6	$\mathcal{D} = \{(z,w) \in \mathbb{C}^3 \times \mathbb{C} : (\operatorname{Im} z_1 - \|w\|^2)^2$ $-(\operatorname{Im} z_1 - \|w\|^2)^2 - (\operatorname{Im} z_3)^2 > 0,$ $\operatorname{Im} z_1 - \|w\|^2 > 0\}$
	$B^1 \times B^2 \times B^2$
	$B^3 \times B^3$
	$B^1 \times B^1 \times B^4$
	$B^2 \times B^5$

Table: The classification so far (Isaev)

Our contribution to the classification

$d(M)$	Contribution
n^2-7	$T_3 \times B^2$
	$T_4 \times B^1$
n^2-8	T_6
	$B^2 \times B^6$
	$B^1 \times B^1 \times B^5$
	$B^1 \times B^1 \times B^1 \times B^2$

Table: The classification so far

Thank you!