Batalin—Vilkovisky Quantisation

An Overview

Contents

Introduction

I will attempt to describe a tool in quantum field theory.

What is the tool good for? What does it do? How does it work?

1 Quantum Field Theory: The Problem

Describe briefly the problem here - quantum field theory is powerful but relies on functional integrals. What are functional integrals?

1.1 Classical Field Theory

In the Lagrangian approach to classical physics, a classical field theory is specified by three pieces of information:

- an underlying *spacetime* manifold M;
- a *space of fields* $\mathcal{F}$, usually given by sections of some sheaf or bundle on M and endowed with the structure of an infinite dimensional manifold; and
- an *action functional* $S\colon \mathcal{F} \to \mathbb{R}$.

The action functional is usually required to satisfy a *locality* criterion, meaning that the action S is of the form

$$S[\phi] = \int_M L(x, j^n_x \phi)$$

for some function L—the *Lagrangian*—depending on the some finite-order jet space of $\mathcal{F}$. Put plainly, this means that S is an integral over M of a function that depends on the field $\phi \in \mathcal{F}$ and only finitely many of its derivatives.

The classical physics of such a system is completely captured by the critical locus of S, that is

$$\mathrm{Crit}(S) := \{\phi \in \mathcal{F} \mid dS[\phi] = 0\}.$$

Using the calculus of variations, in the case that M is without boundary the critical locus of S is the space of solutions to the Euler–Lagrange equations for S.

Example 1.1. The canonical example of a classical field theory is the Newtonian particle in $\mathbb{R}^3$. Here the underlying spacetime manifold is the interval $I = [0, 1]$, the space of fields is $\mathcal{F} := \mathrm{Map}(I, \mathbb{R}^3)$, the space of particle trajectories in $\mathbb{R}^3$ parametrised by $t \in I$, and the action is

$$S[\phi] = \int_0^1 \left(\frac{m}{2} \|\phi'(t)\|^2 - \phi^* V(t) \right) dt$$

for $\phi \in \mathcal{F}$, where $V\colon \mathbb{R}^3 \to \mathbb{R}$ is a smooth potential function and m is a positive real number—the "mass" of the particle represented by ϕ. In this example, a critical point of S corresponds to a solution of the Euler–Lagrange equations

$$m\phi''(t) = -\nabla V(\phi(t)),$$

which is exactly Newton's Second Law for a particle subject to a conservative force.

1.2 From Classical to Quantum

A common technique for passing from a classical to a quantum theory is functional integration. This technique takes the data of a Lagrangian classical field theory as above and purports to extract quantum information from these data.

Suppose we are given a space of fields $\mathcal{F}$ on a spacetime manifold M together with a local action functional S. An *observable* of this system is a function $\mathcal{O}\colon \mathcal{F} \to \mathbb{R}$ and its *expectation value* is given by a weighted integral of $\mathcal{O}$ over the space of fields. Specifically, we have

$$\langle \mathcal{O} \rangle := \frac{1}{Z_S} \int_{\phi \in \mathcal{F}} \mathcal{O}(\phi) e^{iS[\phi]/\hbar} D\phi, \tag{1}$$

where $e^{iS[\phi]/\hbar} D\phi$ is a purported measure on $\mathcal{F}$ and dividing by the *partition function*

$$Z_S := \int_{\phi \in \mathcal{F}} e^{iS[\phi]/\hbar} D\phi$$

makes this into a probability measure on $\mathcal{F}$. The main drawback of this perspective is that it is thoroughly lacking in rigour. The measure theory of the infinite dimensional manifolds that we are dealing with rarely has the properties that we desire and so the path integral must often be used merely as a heuristic tool.

Nonetheless, physicists have developed a whole suite of computational tools built around path integrals that have been wildly successful[1]. The main idea underpinning most of these algorithms is to take integrals in the sandbox environment of the finite dimensional setting, where everything is rigorous and precise, and extrapolate into the infinite dimensional setting. The main challenge of mathematical physicists is to formalise these techniques and couch them in terms of a theory that provides a mathematical justification.

2 Path Integrals

In this section we will briefly discuss the perturbative evaluation of path integrals in quantum field theory. The idea is that by expanding around a non-degenerate critical point of the action, we can define the integral (1) as a formal power series in the expansion parameter $\hbar$. The coefficients in this expansion are given by Gaussian expectation values.

As previously alluded to, this technique relies strongly on some of the properties of finite dimensional Gaussian integrals, so we shall review these first.

2.1 Gaussian Integrals and Beyond...

As previously alluded, the perturbative expansion of path integrals in quantum field theory relies on intuition gleaned from the finite-dimensional setting. Recall that if A is a non-degenerate matrix on $\mathbb{R}^n$, which is equipped with the standard Lebesgue measure $d^n x$ and inner product $(\cdot, \cdot)$, then

$$\int_{\mathbb{R}^n} e^{\frac{i}{2}(x,Ax)} d^n x = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{|\det A|}}\, e^{\frac{i\pi \operatorname{sign} A}{4}},$$

[1]Bar-Natan refers to "the successful religion of path integrals".

with $\operatorname{sign} A$ the signature of A. In order to define expectation values with respect to a Gaussian distribution, the first step is to define the *partition function*

$$Z(J) = \int_{\mathbb{R}^n} e^{\frac{i}{2}(x,Ax)+(J,x)} d^n x = \frac{(\pi\hbar)^{\frac{n}{2}}}{\sqrt{|\det A|}}\, e^{\frac{i\pi \operatorname{sign} A}{4}} \cdot e^{\frac{1}{2}(J,A^{-1}J)}.$$

Then we obtain the expectation value

$$\langle x^{i_1}\cdots x^{i_k}\rangle = \frac{\frac{\partial}{\partial J^{i_1}}\cdots\frac{\partial}{\partial J^{i_k}}\, Z(J)|_{J=0}}{Z(0)} = \frac{\partial}{\partial J^{i_1}}\cdots\frac{\partial}{\partial J^{i_k}} e^{\frac{i}{2}(J,A^{-1}J)}\Big|_{J=0}.$$

This implies the Wick Theorem, which says

$$\langle x^{i_1}\cdots x^{i_k}\rangle = \begin{cases} 0 & \text{if } k \text{ is odd} \\ i^s \sum_{\sigma\in\Sigma_{2s}} \frac{1}{2^s s!}\left(A^{-1}\right)^{i_{\sigma(1)}i_{\sigma(2)}}\cdots\left(A^{-1}\right)^{i_{\sigma(2s-1)}i_{\sigma(2s)}} & \text{if } k = 2s \text{ is even}\end{cases} \tag{2}$$

where Σ_{2s} is the permutation group on $2s$ indices. The Wick Theorem is one of the most crucial ingredients that goes into the perturbative evaluation of path integrals: it is the main result that underpins the technique of Feynman diagrams.

Having detailed the relevant results in finite dimensions, we proceed to the infinite dimensional case. The extension is made by taking the formula (2) verbatim. This makes sense whenever we have an invertible operator A; usually A is a differential operator and $G = A^{-1}$ is its Green function. In the case that $\mathcal{F} = \mathcal{C}^\infty(\Sigma)$ for some manifold Σ and A is a differential operator on $\mathcal{F}$, then we declare

$$\begin{aligned}\langle\phi(x^{i_1})\cdots\phi(x^{i_k})\rangle &:= \frac{\displaystyle\int_{\phi\in\mathcal{F}} e^{\frac{i}{2\hbar}\int_\Sigma \phi A\phi}\phi(x_{i_1})\cdots\phi(x_{i_k})D\phi}{\displaystyle\int_{\phi\in\mathcal{F}} e^{\frac{i}{2\hbar}\int_\Sigma \phi A\phi}D\phi} \\ &:= \begin{cases} 0 & \text{if } k \text{ is odd} \\ (i\hbar)^s \sum_{\sigma\in\Sigma_{2s}} \frac{1}{2^s s!} G(x_{\sigma(1)},x_{\sigma(2)})\cdots G(x_{\sigma(2s-1)},x_{\sigma(2s)}) & \text{if } k = 2s \text{ is even}\end{cases}\end{aligned} \tag{3}$$

where ϕ is a function on Σ, $D\phi$ denotes the "formal" Lebesgue measure on $\mathcal{F}$ and the points $x_{i_1},\ldots,x_{i_k}$ are taken to be distinct. The reason for this latter requirement is that the Green function is often only non-singular on the open configuration space

$$C_2^0(\Sigma) := \Sigma^2 \setminus \Delta.$$

In quantum field theory we are often interested in integrating such expectation values over the configuration space. Making sure that these integrals converge is one of the problems tackled by regularisation and renormalisation.

The next step in the "usual story" of Feynman diagrams is to use the Gaussian moments (3) to calculate expressions of the form

$$\langle\mathcal{O}\rangle := \frac{\displaystyle\int_{\phi\in\mathcal{F}} e^{\frac{i}{2\hbar}S[\phi]}\mathcal{O}(\phi)D\phi}{\displaystyle\int_{\phi\in\mathcal{F}} e^{\frac{i}{2\hbar}S[\phi]}D\phi}. \tag{4}$$

This is done by first expanding $\mathcal{O}$ as a power series in ϕ around a non-degenerate critical point of the action S and then using the expressions for the Gaussian moments. Making full sense of this also requires some regularisation and renormalisation.

2.2 Gauge-fixing Procedures

The above discussion hinged on the fact that the critical point that we expand around is non-degenerate. Unfortunately, this most certainly does not occur when the physical system under consideration has symmetries. The problem of *gauge-fixing* is to make sense of the perturbative expansions in the presence of symmetries.

The BV formalism that is the subject of these notes is a way of solving the problem of gauge-fixing. To better motivate this formalism and to give an introduction to some of the underlying ideas by way of a quasi-historical digression, we first explain some simpler gauge-fixing procedures.

2.2.1 Gauge-fixing: Faddeev—Popov

One of the oldest and most well-known methods of gauge-fixing is the Faddeev—Popov method. We shall explain this by way of an example.

Suppose that our spacetime manifold M is equipped with a vector bundle $E \to M$ whose structure group is the Lie group G (acting in some given representation). The group $\mathcal{G}$ of G-automorphisms of this vector bundle is the *gauge group* of the theory. In this case, the space of fields of our theory is usually the space of sections $\mathcal{F} = \Gamma(M, E)$ and the action is a $\mathcal{G}$-invariant function on $\mathcal{F}$, which can be viewed as a function on the quotient $\mathcal{F}/\mathcal{G}$.

The problem now is to calculate expressions of the form

$$\int_{\phi\in\mathcal{F}} e^{\frac{i}{\hbar}S[\phi]}D\phi \tag{5}$$

via the method of perturbative expansions. The symmetry of S under $\mathcal{G}$ implies that S is degenerate at any critical point and so the perturbative expansion is not well-defined. A natural way to address this problem is to understand (5) as an integral over the coset

$$\frac{1}{\operatorname{vol}\mathcal{G}}\int_{\{\phi\}\in\mathcal{F}/\mathcal{G}} e^{\frac{i}{\hbar}S[\{\phi\}]}D\{\phi\} \tag{6}$$

and this is exactly what the Faddeev—Popov procedure aims to do.

The main idea is the following: choose a smooth function $F\colon \mathcal{F} \to \mathcal{A}$, where $\mathcal{A} = \operatorname{Lie}(\mathcal{G})$ is the Lie algebra of the gauge group, with the property that $F^{-1}(0) \subset \mathcal{F}$ intersects each $\mathcal{G}$-orbit precisely once. The integral (6) is then rewritten as

$$\int_{\phi\in\mathcal{F}} \delta_0(F(\phi)) \cdot J(\phi) \cdot e^{\frac{i}{\hbar}S[\phi]}D\phi. \tag{7}$$

In this expression, the term δ_0 is the delta function at $0 \in \mathcal{A}$, which localises the integral to $F^{-1}(0)$. The term $J(\phi)$ is the determinant of $A(\phi)$, that is $dF(\phi)$ restricted to the vertical tangent space (realised as an endomorphism of $\mathcal{A}$), and ensures invariance under deformations of F.

This can be cast in a more succinct fashion as the integral

$$\int_{\mathcal{F}_{\text{FP}}} e^{\frac{i}{\hbar}S_{\text{FP}}}D\phi Dc D\bar{c} D\lambda \tag{8}$$

over the space of *Faddeev–Popov fields*

$$\mathcal{F}_{\text{FP}} := \mathcal{F} \oplus \mathcal{A}[1] \oplus \mathcal{A}^*[-1] \oplus \mathcal{A}^* \ni (\phi, c, \bar{c}, \lambda).$$

In this expression, the Faddeev–Popov action

$$S_{\text{FP}}[\phi, c, \bar{c}, \lambda] = S[\phi] + \langle \lambda, F(\phi) \rangle + \langle \bar{c}, A(\phi) c \rangle$$

is non-degenerate at its critical point(s) and so allows for a well-defined perturbative expansion. In this discussion, F is called a *gauge-fixing function* and the condition $F = 0$ is the *gauge-fixing*.

Example 2.1. As an example of the Faddeev–Popov method, consider the case where we have a finite-dimensional space of fields $\mathcal{F}$ acted on by a finite-dimensional Lie group G. Suppose that we also have a G-invariant action functional $S\colon \mathcal{F} \to \mathbb{R}$.

The action of G on $\mathcal{F}$ induces a Lie algebra homomorphism $\mathfrak{g} := \text{Lie}(G) \to \mathfrak{X}(\mathcal{F})$. In terms of a basis e_α for $\mathfrak{g}$, we have

$$e_\alpha \longmapsto v^i_\alpha(x) \frac{\partial}{\partial x^i} \in \mathfrak{X}(\mathcal{F}). \tag{9}$$

To make sense of the expression (6), we must pick an appropriate gauge-fixing function $F\colon \mathcal{F} \to \mathfrak{g}$. Then (7) becomes

$$\int_{x \in \mathcal{F}} e^{\frac{i}{\hbar} S} \prod_{i,\alpha} \delta(F^\alpha(x)) \cdot \det \left(v^j_\alpha(x) \frac{\partial}{\partial x^j} F^\beta(x) \right) Dx^i.$$

Introducing the Grassmann variables c^α and $\bar{c}_\alpha$ of $\mathbb{Z}$-degree 1 and -1 respectively and the variable λ_α of $\mathbb{Z}$-degree 0, we can write this last integral as

$$\int_{\mathcal{F}_{\text{FP}}} e^{\frac{i}{\hbar} S_{\text{FP}}} \prod_{i,\alpha} Dx^i Dc^\alpha D\bar{c}_\alpha D\lambda_\alpha$$

where

$$S_{\text{FP}} = S(x) + \lambda_\alpha F^\alpha(x) + \bar{c}_\alpha \frac{\partial F^\alpha}{\partial x^i} v^i_\beta(x) c^\beta$$

and the space of Faddeev–Popov fields is $\mathcal{F}_{\text{FP}} = \mathcal{F} \oplus \mathfrak{g}[1] \oplus \mathfrak{g}^*[-1] \oplus \mathfrak{g}^*$.

2.2.2 Gauge-fixing: BRST

A second, more refined method of treating the problem of gauge fixing is the BRST formalism. The framework of this formalism is as follows. We embed the manifold of classical fields $\mathcal{F}$ into the 0-th degree of a $\mathbb{Z}$-graded manifold $\mathcal{F}_{\text{BRST}}$. The $\mathbb{Z}$-grading on $\mathcal{F}_{\text{BRST}}$ is known in physics parlance as the *ghost number*.

The $\mathbb{Z}$-graded manifold is equipped with the extra information of a cohomological vector field. This is a derivation Q on $\text{Fun}(\mathcal{F}_{\text{BRST}})$ that raises the $\mathbb{Z}$-degree (ghost number) by 1 and squares to zero; i.e. $Q^2 = 0$. The $\mathbb{Z}$-graded manifold $\mathcal{F}_{\text{BRST}}$ is assumed to be equipped with a Q-invariant measure μ, so that in particular

$$\int_{\mathcal{F}_{\text{BRST}}} (Qf)\mu = 0$$

for every $f \in \text{Fun}(\mathcal{F}_{\text{BRST}})$. The data $(\mathcal{F}_{\text{BRST}}, Q)$ of the BRST formalism is related to the data of the classical theory by the conditions

- the classical action functional is a BRST cocycle, i.e. $QS = 0$; and
- $\mathcal{F}_{\text{BRST}}$ is a resolution of $\mathcal{F}/\mathcal{G}$ in the sense that there is an isomorphism

$$H^0_Q(\text{Fun}(\mathcal{F}_{\text{BRST}})) \cong \text{Fun}(\mathcal{F}/\mathcal{G}).$$

Remark 2.2. This latter point is particularly interesting as it leads to the interpretation of the BRST procedure as taking the derived quotient of a space by a Lie algebra action. This point of view is currently being clarified by a number of researchers in the area.

Gauge-fixing in the BRST formalism proceeds as follows. First, we choose a *gauge-fixing fermion*, which is a function $\Psi \in \text{Fun}(\mathcal{F}_{\text{BRST}})$ with $\mathbb{Z}$-degree -1. Then since the measure μ is assumed to be Q-invariant, we have

$$\int_{\mathcal{F}_{\text{BRST}}} e^{\frac{i}{\hbar}S}\mu = \int_{\mathcal{F}_{\text{BRST}}} e^{\frac{i}{\hbar}(S+Q\Psi)}\mu.$$

The idea is to be cunning and choose a gauge-fixing fermion so that the right hand side of the above equation allows for a perturbative expansion. The modified action

$$S_{\text{BRST}} = S + Q\Psi$$

is called the BRST action. In this setting, we can prove (at least formally) that

1. the expectation values of Q-closed functions are gauge-fixing independent; and
2. the expectation values of Q-exact functions vanish.

We have therefore interpreted the path integral in the BRST formalism as some sort of cohomological operation; this observation will shape our view of the BV formalism.

Example 2.3. We continue Example 2.1 in the setting of the BRST formalism. In this case,

$$\mathcal{F}_{\text{BRST}} = \mathcal{F} \oplus \mathfrak{g}[1]$$

and in coordinates (x^i, c^α) on $\mathcal{F}_{\text{BRST}}$ the cohomological vector field is given by

$$Q = -c^\alpha v^i_\alpha(x)\frac{\partial}{\partial x^i} + \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma \frac{\partial}{\partial c^\alpha}$$

where the $f^\alpha_{\beta\gamma}$ are the structure constants of $\mathfrak{g}$ in the basis e_α (dual to c^α). The condition

$$QS = 0$$

encodes the (infinitesimal) G-invariance of S and the condition $Q^2 = 0$ encodes the Jacobi identity for the $f^\alpha_{\beta\gamma}$ as well as the fact that (9) is a Lie algebra homomorphism. The idea is that the data of the G-symmetry of the classical system is encoded in the cohomological vector field Q.

In order to perform gauge-fixing in this example, we need to write down a gauge-fixing fermion. Since a gauge-fixing fermion must necessarily have $\mathbb{Z}$-degree -1 (and the coordinates x^i and c^α have non-negative degrees), we must extend the space $\mathcal{F}_{\text{BRST}}$ of BRST fields. The idea is to do this in such a way as to not change the Q-cohomology. For example, if we take

$$\mathcal{F}_{\text{BRST}} = \mathcal{F} \oplus \mathfrak{g}[1] \oplus \mathfrak{g}^*[-1] \oplus \mathfrak{g}^* \tag{10}$$

with coordinates $(x^i, c^\alpha, \bar{c}_\alpha, \lambda_\alpha)$, and

$$Q = -c^\alpha v^i_\alpha(x)\frac{\partial}{\partial x^i} + \frac{1}{2}f^\alpha_{\beta\gamma}c^\beta c^\gamma\frac{\partial}{\partial c^\alpha} + \lambda_\alpha\frac{\partial}{\partial \bar{c}_\alpha} \tag{11}$$

then the Q-cohomology doesn't change but we can now write down the gauge-fixing fermion

$$\Psi = \bar{c}_\alpha F^\alpha(x),$$

where F is the gauge-fixing function that we chose in Example 2.1. The BRST action is then

$$S_{\text{BRST}} = S + Q\Psi = S(x) + \lambda_\alpha F^\alpha(x) + \bar{c}_\alpha\frac{\partial F^\alpha}{\partial x^i}v^i_\beta(x)c^\beta$$

which agrees with our result from the Faddeev–Popov method.

Remark 2.4. In order to write down the gauge-fixing fermion Ψ in the above example, we had to add the vector space $\mathfrak{g}^*[-1]\oplus\mathfrak{g}^*$ to our space of BRST fields in (10). As we shall see below, we can view this extra term as the $\mathbb{Z}$-graded manifold $T[1](\mathfrak{g}^*[-1])$, whose ring of functions is $\Omega^\bullet(\mathfrak{g}^*[-1])$. The term that we added to the cohomological vector field Q in (11) corresponds to the de Rham differential. Since $\mathfrak{g}^*[-1]$ is contractible, this additional term does not change the Q-cohomology of $\mathcal{F}_{\text{BRST}}$.

3 Batalin—Vilkovisky Quantisation

The Batalin—Vilkovisky formalism is a body of ideas for treating the problem of gauge-fixing using homological algebra. In its first guise, which we shall describe presently, it may be viewed as a far-reaching generalisation of the BRST formalism.

The essential ingredient of the BV approach is the *space of BV fields* $\mathcal{F}_{\text{BV}} = T^*[-1]\mathcal{F}_{\text{BRST}}$. As the shifted cotangent bundle of a $\mathbb{Z}$-graded manifold, $\mathcal{F}_{\text{BV}}$ is naturally an odd symplectic $\mathbb{Z}$-graded manifold and this structure is key to understanding this guise of the BV formalism. It is therefore paramount at this point to review the crucial definitions.

3.1 A Digression on BV Manifolds

Definition 3.1. A $\mathbb{Z}$-*graded manifold* is a locally ringed space $(\mathcal{M}, \mathcal{O})$ whose ring of functions is locally isomorphic to

$$(U, \mathcal{C}^\infty(U)\otimes S(V^*)),$$

where U is an open subset of $\mathbb{R}^n$ and V is a $\mathbb{Z}$-graded vector space.

Remark 3.2. Recall that for a $\mathbb{Z}$-graded vector space V, the *dual* is the graded vector space given in degree i by V^*_{-i}. The *shift* $V[k]$ of V by k is the graded vector space given in degree i by V_{i+k}.

The *symmetric algebra* $S(W)$ generated by a graded vector space W is the quotient of the tensor algebra on W by the tensor ideal generated by elements of the form

$$a\otimes b - (-1)^{|a||b|}b\otimes a$$

where $a, b \in W$ are homogeneous of degree $|a|$ and $|b|$ respectively.

Alternatively, we may view a $\mathbb{Z}$-graded manifold $\mathcal{M}$ as a sum of vector bundles

$$\mathcal{M} = \bigoplus_{k\in\mathbb{Z}} \mathcal{M}^k$$

over a smooth manifold $\mathcal{M}_0$—the *body* of $\mathcal{M}$—only finitely many of which are non-zero. The ring of functions on $\mathcal{M}$ is the graded ring of sections

$$\mathrm{Fun}(\mathcal{M}) := \Gamma(\mathcal{M}_0, S(\mathcal{M}^*_{\mathrm{even}})) \otimes_{\mathcal{C}^\infty(\mathcal{M}_0)} \Gamma(\mathcal{M}_0, \Lambda(\mathcal{M}^*_{\mathrm{odd}})),$$

where $\mathcal{M}_{\mathrm{even}} = \bigoplus_{k\in\mathbb{Z}} \mathcal{M}^{2k}$ and $\mathcal{M}_{\mathrm{odd}} = \bigoplus_{k\in\mathbb{Z}} \mathcal{M}^{2k+1}$ are the even and odd parts of $\mathcal{M}$, and S and Λ denote the symmetric and exterior algebras respectively.

There are two important graded manifolds that we can associate to a given graded manifold $\mathcal{M}$. The first of these is the *shifted tangent bundle* $T[1]\mathcal{M}$. As a graded manifold, $T[1]\mathcal{M}$ is represented by the sum of vector bundles $\bigoplus_{k\in\mathbb{Z}}(T[1]\mathcal{M})^k$ over $\mathcal{M}_0$, where

$$(T[1]\mathcal{M})^k = \begin{cases} T\mathcal{M}_0 \oplus \mathcal{M}^{-1} \oplus \mathcal{M}^0 & \text{for } k = -1 \\ \mathcal{M}^k \oplus \mathcal{M}^{k+1} & \text{for } k \neq -1. \end{cases}$$

We define the ring of *differential forms* on $\mathcal{M}$ to be functions on the shifted tangent bundle

$$\Omega^\bullet(\mathcal{M}) := \mathrm{Fun}(T[1]\mathcal{M}).$$

A k-form on $\mathcal{M}$ is thus understood as a degree k element of $\mathrm{Fun}(T[1]\mathcal{M})$.

Remark 3.3. Contrast this definition to that of *vector fields* on $\mathcal{M}$, which are given by derivations of the algebra $\mathrm{Fun}(\mathcal{M})$. In particular, we can talk about vector fields on the shifted tangent bundle. A particularly important example of a vector field on $T[1]\mathcal{M}$ is given by the *de Rham differential*. Picking local coordinates x^i on $\mathcal{M}$, we get coordinates dx^i on the fibre (considering $T[1]\mathcal{M}$ as a bundle over $\mathcal{M}$). Note that $|dx^i| = |x^i| + 1$ for each i. The de Rham differential is then given by the vector field

$$\delta := dx^i \frac{\partial}{\partial x^i}$$

acting on $\mathrm{Fun}(T[1]\mathcal{M})$. The de Rham differential raises the degree of functions on $T[1]\mathcal{M}$ by 1 and squares to zero.

We note that in the case that $\mathcal{M} = M$ is an ordinary manifold viewed as a graded manifold in the obvious way, then defining $\Omega^\bullet(M) := \mathrm{Fun}(T[1]M)$ and $d_{\mathrm{dR}} = \delta$ as above coincides exactly with the usual construction of the de Rham complex on M.

Another important notion is the *shifted cotangent bundle* $T^*[-1]\mathcal{M}$ of a graded manifold $\mathcal{M}$. This is a graded bundle represented by the bundle $\bigoplus_{k\in\mathbb{Z}}(T^*[-1]\mathcal{M})^k$ over $\mathcal{M}_0$, where

$$(T^*[-1]\mathcal{M})^k = \begin{cases} T^*\mathcal{M}_0 \oplus \mathcal{M}^1 \oplus (\mathcal{M}^0)^* & \text{for } k = 1 \\ \mathcal{M}^k \oplus (\mathcal{M}^{1-k})^* & \text{for } k \neq 1. \end{cases}$$

If we choose coordinates x^i on $\mathcal{M}$ and ξ_i are coordinates in the fibre of $T^*[-1]\mathcal{M}$ (considered as a bundle over $\mathcal{M}$) then the degrees satisfy $|x^i| + |\xi_i| = -1$. We define the ring of *polyvector fields* on $\mathcal{M}$ to be

$$\mathfrak{V}^\bullet(\mathcal{M}) := \mathrm{Fun}(T^*[-1]\mathcal{M}).$$

As was the case with differential forms, this construction coincides with the usual one when $\mathcal{M} = M$ is an ordinary smooth manifold.

Definition 3.4. An *odd symplectic manifold* is a graded manifold $\mathcal{N}$ that is equipped with a 2-form $\omega \in \Omega^\bullet(\mathcal{N})$ of total degree 1 and de Rham degree 2. We also require that $\mathcal{N}$ has an open covering $\{U_\alpha\}$ such that in each U_α there is a *Darboux coordinate system* $(x^i_\alpha, \xi_{(\alpha)\,i})$ in which ω is given by

$$\omega = \sum_i (-1)^{|x^i_{(\alpha)}|} \delta x^i_{(\alpha)} \delta \xi_{(\alpha)\,i},$$

and for which the coordinate transition maps $\phi_{\alpha\beta}$ between Darboux coordinate systems are symplectomorphisms, that is $\phi^*_{\alpha\beta}\omega = \omega$.

The shifted cotangent bundle $T^*[-1]\mathcal{M}$ is the canonical example of an odd symplectic manifold, with odd symplectic form given in local coordinates (x^i, ξ_i) by

$$\omega = \sum_i (-1)^{|x^i|} \delta x^i \delta \xi_i. \tag{12}$$

By definition, every odd symplectic manifold is locally equivalent to some $T^*[-1]\mathcal{N}$, and in fact a stronger global result holds

Theorem 3.5 (A. Schwarz). *Every odd symplectic manifold $\mathcal{M}$ is symplectomorphic to an odd symplectic manifold of the form $T^*[-1]\mathcal{N}$, where $\mathcal{N}$ may be chosen to be concentrated in even degrees.*

In the same manner that ordinary symplectic forms induce a Poisson bracket on functions, an odd symplectic form on $\mathcal{M}$ induces a Gerstenhaber bracket on the the algebra of functions $\mathrm{Fun}(\mathcal{M})$. In the case of the canonical odd symplectic form (12), the bracket is given by

$$\{f, g\} := \sum_i f \left[\frac{\overleftarrow{\partial}}{\partial x^i} \frac{\overrightarrow{\partial}}{\partial \xi_i} - \frac{\overleftarrow{\partial}}{\partial \xi_i} \frac{\overrightarrow{\partial}}{\partial x^i} \right] g \tag{13}$$

for $f, g \in \mathrm{Fun}(\mathcal{M})$.

We recall

Definition 3.6. A *Gerstenhaber* or *odd Poisson algebra* is a $\mathbb{Z}$-graded commutative algebra A equipped with a bilinear map $\{\bullet, \bullet\} \colon A \otimes A \to A$ such that for homogeneous $a, b, c \in A$

1. $|\{a, b\}| = |a| + |b| + 1$,
2. $\{a, b\} = -(-1)^{(|a|+1)(|b|+1)}\{b, a\}$,
3. $(-1)^{(|a|+1)(|c|+1)}\{a, \{b, c\}\}$ + cyclic permutations of $a, b, c = 0$,
4. $\{ab, c\} = a\{b, c\} + (-1)^{|b|(|c|+1)}\{a, c\}b$, and
5. $\{a, bc\} = \{a, b\}c + (-1)^{|b|(|a|+1)}b\{a, c\}$.

Therefore $\mathrm{Fun}(\mathcal{M})$ with the bracket $\{\bullet, \bullet\}$ of (13) is a Gerstenhaber algebra.

3.1.1 BV Manifolds and the BV Laplacian

Definition 3.7. A *BV manifold*[2] is an odd symplectic manifold $(\mathcal{M}, \omega)$ endowed with a measure μ that is compatible with the odd symplectic form in the following sense. In the Darboux coordinates $(x^i_{(\alpha)}, \xi_{(\alpha)\,i})$ on the chart U_α the measure μ coincides with the coordinate Berezin measure. That is

$$\mu = \prod_i Dx^i_{(\alpha)} \cdot D\xi_{(\alpha)\,i}$$

on U_α. We moreover require that the coordinate transition functions $\phi_{\alpha\beta}$ between Darboux charts are measure-preserving.

In the setting of a shifted cotangent bundle $T^*[-1]\mathcal{M}$ equipped with the canonical odd symplectic form ω. We view $T^*[-1]\mathcal{M}$ as a BV manifold equipped with the measure μ given in the local coordinates (x^i, ξ_i) by

$$\mu = \prod_i Dx^i D\xi_i, \tag{14}$$

which is clearly compatible with the odd symplectic form (12).

The measure μ on a BV manifold $\mathcal{M}$ defines a *divergence operator* on vector fields. Given a vector field $X \in \mathfrak{X}(\mathcal{M})$ (that is, a derivation $X\colon \operatorname{Fun}(\mathcal{M}) \to \operatorname{Fun}(\mathcal{M})$), we define its *divergence* (with respect to the measure μ) as the function $\operatorname{div}_\mu X \in \operatorname{Fun}(\mathcal{M})$ such that

$$\int_{\mathcal{M}} X(f)\mu = \int_{\mathcal{M}} \operatorname{div}_\mu X \cdot f\mu$$

for all $f \in \operatorname{Fun}(\mathcal{M})$. We then define the *BV Laplacian* as the operator

$$\Delta_\mu\colon \operatorname{Fun}(\mathcal{M}) \longrightarrow \operatorname{Fun}(\mathcal{M})$$

that sends

$$f \longmapsto \Delta_\mu f := \frac{(-1)^{|f|}}{2} \operatorname{div}_\mu(\{f, \bullet\}).$$

In Darboux coordinates (x^i, ξ_i) on $\mathcal{M}$ the BV Laplacian may be expressed as

$$\Delta_\mu = \sum_i (-1)^{|x^i|} \frac{\partial}{\partial x^i} \frac{\partial}{\partial \xi_i}. \tag{15}$$

There are various relations that Δ_μ satisfies, amongst which are

$$\{f, g\} = (-1)^{|f|}\Delta_\mu(fg) - (-1)^{|f|}\Delta_\mu(f)g - f\Delta_\mu(g)$$

and

$$\Delta_\mu(\{f, g\}) = \{\Delta_\mu(f), g\} + (-1)^{|f|+1}\{f, \Delta_\mu g\}$$

for all homogeneous $f, g \in \operatorname{Fun}(\mathcal{M})$.

[2] this is commonly referred to as an SP-manifold in the literature.

3.1.2 Lagrangian Submanifolds and Integration

Suppose we are given the BV manifold $(\mathcal{M}, \omega, \mu)$ (or, more generally, an odd symplectic manifold $(\mathcal{M}, \omega)$). A *Lagrangian submanifolds* of $\mathcal{M}$, as in ordinary symplectic geometry, is a submanifold $\mathcal{L} \subset \mathcal{M}$ such that $\omega|_{\mathcal{L}} = 0$ and $\dim \mathcal{L} = \frac{1}{2} \dim \mathcal{M}$.

In the case that $\mathcal{M} = T^*[-1]\mathcal{F}$ is a shifted cotangent bundle there is an important class of Lagrangian submanifolds defined via *gauge-fixing functions*. Suppose that $\mathcal{M} = T^*[-1]\mathcal{F}$ is given in local Darboux coordinates by (x^i, ξ_i). Then $\mathcal{F} \subset T^*[-1]\mathcal{F}$ is itself a Lagrangian submanifold. More generally, given a *gauge-fixing function* $\Psi \in \mathrm{Fun}(\mathcal{F})$ of degree -1 we may associate to it the Lagrangian submanifold

$$\mathcal{L}_\Psi := \left\{ (x^i, \xi_i) \;\middle|\; \xi_i = -\frac{\partial}{\partial x^i} \Psi(x) \right\}.$$

An important feature of BV manifolds is that every Lagrangian submanifold has an induced measure. If $\mathcal{L} \subset \mathcal{M}$ is a Lagrangian submanifold of the BV manifold $\mathcal{M}$ with measure μ, we denote the induced measure on $\mathcal{L}$ by $\sqrt{\mu}_{\mathcal{L}}$. The following "graded" version of the Stokes theorem is really the key to the BV formalism:

Theorem 3.8 (Batalin—Vilkovisky, Schwarz). *Let $(\mathcal{M}, \omega, \mu)$ be a BV manifold. Let $\mathcal{L}, \mathcal{L}' \subset \mathcal{M}$ be Lagrangian submanifolds with induced measures $\sqrt{\mu}_{\mathcal{L}}$ and $\sqrt{\mu}_{\mathcal{L}'}$ respectively. Then*

1. *if $f \in \mathrm{Fun}(\mathcal{M})$ is Δ_μ-closed and $\mathcal{L}$ and $\mathcal{L}'$ are homologous (in $\mathcal{M}_0$) then*

$$\int_{\mathcal{L}} f \sqrt{\mu}_{\mathcal{L}} = \int_{\mathcal{L}'} f \sqrt{\mu}_{\mathcal{L}'}.$$

   That is, the integral of a Δ_μ-closed function over a Lagrangian submanifold $\mathcal{L}$ is invariant under infinitesimal deformations of $\mathcal{L}$; and

2. *if $f \in \mathrm{Fun}(\mathcal{M})$ is Δ_μ-exact then*

$$\int_{\mathcal{L}} f \sqrt{\mu}_{\mathcal{L}} = 0.$$

3.1.3 The Master Equation

From now on, we consider the ring $\mathrm{Fun}(\mathcal{M})[[\hbar]]$ of formal power series in $\hbar$. Let $(\mathcal{M}, \omega, \mu)$ be a BV manifold and take a function of the form

$$S = S_0 + S_1 \hbar + S_2 \hbar^2 + \cdots \in \mathrm{Fun}(\mathcal{M})[[\hbar]] \tag{16}$$

that is regular in $\hbar$. We say that S satisfies the *quantum master equation* (and hence is a *quantum master function*) if

$$\Delta_\mu e^{\frac{i}{\hbar} S} = 0 \tag{17}$$

or, equivalently

$$\{S, S\} + i\hbar \Delta_\mu S = 0, \tag{18}$$

recalling the bracket $\{\bullet, \bullet\}$ of (13).

3.2 BV Quantisation

Having reviewed the necessary ingredients, we now describe the BV formalism. The central idea of the BV formalism is to take a classical field theory, specified by the data $(M, \mathcal{F}_{\text{cl}}, S_{\text{cl}})$, and construct from it a BV manifold. Gauge-fixing is then carried out by integrating over Lagrangian submanifolds via Theorem 3.8, as we describe presently.

To construct the required BV manifold, we embed the space of fields $\mathcal{F}$ of the classical theory into the shifted cotangent bundle

$$\mathcal{F}_{\text{BV}} := T^*[-1]\mathcal{F}_{\text{BRST}}. \tag{19}$$

That is, we first carry out the BRST procedure to obtain the space $\mathcal{F}_{\text{BRST}}$ of BRST fields and then we take its shifted cotangent bundle. Note that $\mathcal{F}_{\text{BV}}$ is canonically a BV manifold, equipped with odd symplectic form

$$\omega = \sum_i (-1)^{|\Phi^a|} \delta\Phi^a \delta\Phi^\dagger_a \tag{20}$$

and compatible measure

$$\mu = \prod_i D\Phi^a D\Phi^\dagger_a. \tag{21}$$

In these expressions, the Φ^a are coordinates on $\mathcal{F}_{\text{BRST}}$ (the *fields*) and the $\Phi^\dagger_a$ are the corresponding coordinates on the fibres of $T^*[-1]\mathcal{F}_{\text{BRST}}$ (the *anti-fields*). Note that the BV Laplacian is given simply by

$$\Delta = \sum_a (-1)^{|\Phi^a|} \frac{\partial}{\partial\Phi^a}\frac{\partial}{\partial\Phi^\dagger_a}. \tag{22}$$

The next thing that we need is a quantum master function S that is consistent with the classical action S_{cl} in the following sense

$$S_0|_{\mathcal{F}_{\text{BRST}}} = S_{\text{cl}}.$$

Gauge-fixing in the BV formalism is carried out by integrating over Lagrangian submanifolds, that is

$$\int_{\mathcal{F}_{\text{cl}}/\mathcal{G}} e^{\frac{i}{\hbar}S_{\text{cl}}} := \int_{\mathcal{L}} e^{\frac{i}{\hbar}S}\sqrt{\mu}_{\mathcal{L}} \tag{23}$$

for $\mathcal{L} \subset \mathcal{F}_{\text{BV}}$ a Lagrangian submanifold. Choosing such a Lagrangian submanifold therefore corresponds to a choice of gauge-fixing.

In order to define expectation values $\langle\mathcal{O}\rangle$ in the BV formalism, we make use of the graded Stokes Theorem 3.8. Recall that the expectation value of $\mathcal{O} \in \text{Fun}(\mathcal{F}_{\text{cl}})$ is defined by the expression (4), so that in the BV formalism

$$\langle\mathcal{O}\rangle := \frac{\int_{\mathcal{L}} e^{\frac{i}{\hbar}S}\mathcal{O}\sqrt{\mu}_{\mathcal{L}}}{\int_{\mathcal{L}} e^{\frac{i}{\hbar}S}\sqrt{\mu}_{\mathcal{L}}} \tag{24}$$

for some choice of Lagrangian $\mathcal{L} \subset \mathcal{F}_{\text{BV}}$. Taking expectation values in the BV formalism is then done as follows. Take a function $\mathcal{O} \in \text{Fun}(\mathcal{F}_{\text{BV}})$, then

- if $\Delta\mathcal{O} = 0$ we have that $\langle\mathcal{O}\rangle$ is independent of continuous deformations of the Lagrangian submanifold $\mathcal{L}$; and

- if $\mathcal{O}$ is Δ-exact then $\langle\mathcal{O}\rangle = 0$.

Both of these points follow directly from Theorem 3.8 and the fact that S satisfies the quantum master equation (18). Essentially what is being said here is that the Δ-cohomology of $\mathrm{Fun}(\mathcal{F}_{\mathrm{BV}})$ controls the gauge-fixing of path integrals. In particular, if $\mathcal{O}$ is a Δ-closed function, then its gauge-fixed expectation value is independent under (infinitesimal) deformations of the choice of gauge.

The upshot of all of this is that the BV formalism provides us with a nice way to control gauge-fixing via homological algebra. Notice that the perturbative evaluation of the path integrals appearing in, for example, (24) still requires some regularity condition on the classical master function S. Namely, S must be non-degenerate for some choice of Lagrangian $\mathcal{L} \subset \mathcal{F}_{\mathrm{BV}}$. The perturbative expansion then follows the standard procedure after restricting everything in sight to $\mathcal{L}$.

3.3 A Remark: the Classical Master Equation

The most subtle aspect of using the BV formalism is constructing the quantum master function. The rest of the structure that we need—namely that of a BV manifold—is provided almost entirely by the construction outlined above.

A standard approach to constructing the quantum master function S is to build S order-by-order in $\hbar$ via the power series expansion (16). Supposing that $S = S_0 + S_1\hbar + \cdots$ is a quantum master function, then writing the corresponding terms of the quantum master equation order-by-order in $\hbar$ gives

$$\begin{aligned}\{S_0, S_0\} &= 0 \\ \{S_0, S_1\} &= -i\Delta S_0 \\ \{S_0, S_2\} &= -\tfrac{1}{2}\{S_1, S_1\} - i\Delta S_1\end{aligned}$$

and so on. The first step is to therefore extend the classical action $S_{\mathrm{cl}} \in \mathrm{Fun}(\mathcal{F}_{\mathrm{cl}}) \subset \mathrm{Fun}(\mathcal{F}_{\mathrm{BRST}})$ to a solution S_0 of the *classical master equation*

$$\{S_0, S_0\} = 0 \tag{25}$$

in such a way that $S_0|_{\mathcal{F}_{\mathrm{BRST}}} = S_{\mathrm{cl}}$. Assuming that we have a solution to the classical master equation, the derivation $Q := \{S_0, \bullet\}$ on $\mathrm{Fun}(\mathcal{F}_{\mathrm{BV}})$ squares to zero and so we can consider its cohomology groups $H^\bullet_Q(\mathrm{Fun}(\mathcal{F}_{\mathrm{BV}}))$ (where the grading is by $\mathbb{Z}$-degree). In order to construct a quantum master function to order $\hbar$, we therefore require $[-i\Delta S_0] = 0 \in H^1_Q(\mathrm{Fun}(\mathcal{F}_{\mathrm{BV}}))$. To extend to order $\hbar^2$, we require that the Q-cohomology class of $-\frac{1}{2}\{S_1, S_1\} - i\Delta S_1$ is trivial, and so on. In particular, a solution to the classical master equation extends to a quantum master function in the case that $H^1_Q(\mathrm{Fun}(\mathcal{F}_{\mathrm{BV}})) = 0$, so we can view this cohomology group as an obstruction group for quantisation.

3.4 A First Example

We now continue our old example (see Examples 2.1 and 2.3) in the setting of the BV formalism. We have already carried out the BRST formalism to obtain

$$\mathcal{F}_{\mathrm{BRST}} = \mathcal{F} \oplus \mathfrak{g}[1] \oplus \mathfrak{g}^*[-1] \oplus \mathfrak{g}^*$$

with coordinates $(x^i, c^\alpha, \bar{c}_\alpha, \lambda_\alpha)$, and

$$Q = -c^\alpha v^i_\alpha(x)\frac{\partial}{\partial x^i} + \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma \frac{\partial}{\partial c^\alpha} + \lambda_\alpha \frac{\partial}{\partial \bar{c}_\alpha}.$$

According to (19), the space of BV fields is

$$\mathcal{F}_{\text{BV}} = T^*[-1]\mathcal{F}_{\text{BRST}} = T^*[-1](\mathcal{F} \oplus \mathfrak{g}[1] \oplus \mathfrak{g}^*[-1] \oplus \mathfrak{g}^*).$$

This is a BV manifold with coordinates $\Phi^a := (x^i, c^\alpha, \bar{c}_\alpha, \lambda_\alpha)$ on the base (the ***fields***) and coordinates $\Phi^\dagger_a := (x^\dagger_i, c^\dagger_\alpha, \bar{c}^{\dagger\alpha}, \lambda^{\dagger\alpha})$ on the fibres (the ***anti-fields***). The odd symplectic form ω, compatible measure μ and BV Laplacian Δ are given by (20), (21) and (22) respectively.

One way to construct the quantum master function in this setting is via the expression

$$S = S_{\text{cl}} - Q(\Phi^\alpha)\Phi^\dagger_\alpha,$$

where Q is the BRST operator of (11). This gives

$$S = S_{\text{cl}}(x) - x^\dagger_i v^i_\alpha(x) c^\alpha + \frac{1}{2} f^\alpha_{\beta\gamma} c^\dagger_\alpha c^\beta c^\gamma - \bar{c}^{\dagger\alpha}\lambda_\alpha.$$

The fact that the above S is a quantum master function can be deduced from $QS_{\text{cl}} = 0$, $Q^2 = 0$ and the assumption $\operatorname{div}_\mu Q = 0$ (which me make for simplicity).

If we now choose the gauge-fixing function

$$\Psi = \bar{c}_\alpha F^\alpha(x)$$

as in Example 2.3, we have the associated Lagrangian submanifold

$$\begin{aligned}\mathcal{L}_\Psi &= \left\{ (\Phi^a, \Phi^\dagger_a) \in \mathcal{F}_{\text{BV}} \;\middle|\; \Phi^\dagger_a = -\frac{\partial}{\partial \Phi^a}\Psi \right\} \\ &= \left\{ (x^i, c^\alpha, \bar{c}_\alpha, \lambda_\alpha, x^\dagger_i, c^\dagger_\alpha, \bar{c}^{\dagger\alpha}, \lambda^{\dagger\alpha}) \;\middle|\; x^\dagger_i = \frac{\partial F^\alpha}{\partial x^i}(x)\bar{c}_\alpha,\ c^\dagger_\alpha = 0,\ \bar{c}^{\dagger\alpha} = -F^\alpha(x),\ \text{and } \lambda^{\dagger\alpha} = 0 \right\}.\end{aligned}$$

In this case, the integral

$$\int_{\mathcal{L}_\Psi} e^{\frac{i}{\hbar}S}\sqrt{\mu}_{\mathcal{L}_\Psi} = \int_{\mathcal{F}_{\text{FP}}} \exp\left(\frac{i}{\hbar}\left[S_{\text{cl}}(x) + \lambda_\alpha F^\alpha(x) + \bar{c}_\alpha \frac{\partial F^\alpha}{\partial x^i} v^i_\beta(x) c^\beta \right]\right) \prod_{i,\alpha} Dx^i Dc^\alpha D\bar{c}_\alpha D\lambda_\alpha$$

coincides with our results from the Faddeev–Popov and BRST procedures.

3.5 A Modern Perspective

A more modern perspective on the BV formalism follows the ideas of Costello and Gwilliam. This viewpoint interprets the BV formalism as a two-step procedure,

1. the *classical* BV formalism being the application of derived geometry to compute the derived critical locus of a classical action; and
2. the *quantum* BV formalism being a way of computing expectation values of functions on this derived critical locus via homological algebra.

Since we do not have the tools to properly treat the first part of this process, we will focus primarily on the second part.

3.5.1 A Sketch of the Derived Critical Locus

Suppose we have a classical field theory specified by the space of fields $\mathcal{F}_{\text{cl}}$ and action functional S. For simplicity, we take $\mathcal{F}_{\text{cL}} = M$ to be a smooth finite-dimensional manifold. As we have already remarked, the classical physics of this system is controlled by the critical locus

$$\text{Crit}(S) := \{x \in M \mid dS(x) = 0\} = \text{graph}(dS) \times_{T^*M} M,$$

where the latter equality reflects the fact that $\text{Crit}(S)$ is given by the intersection of the graph of dS and the zero section inside T^*M. In terms of sheaves of functions,

$$\mathcal{O}(\text{Crit}(S)) = \mathcal{O}(\text{graph}(dS)) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M).$$

One way of understanding the derived critical locus, which we will not explain in sufficient detail, is to replace the tensor product above with the "homologically correct" tensor product. The idea is that while the above expression captures the data of the naive intersection $\text{graph}(dS) \times_{T^*M} M$, the *derived* tensor product in

$$\mathcal{O}(\text{dCrit}(S)) = \mathcal{O}(\text{graph}(dS)) \otimes^{\mathbb{L}}_{\mathcal{O}(T^*M)} \mathcal{O}(M)$$

also captures more refined information.

In order to calculate the derived tensor product, we must first pick a resolution of $\mathcal{O}(\text{graph}(dS))$ over $\mathcal{O}(T^*M)$. Avoiding technical questions related to this (i.e. what model category structure are we using?), we note that a particular choice of resolution gives

$$\mathcal{O}(\text{dCrit}(S)) = \Lambda^{\dim M} T_M \longrightarrow \cdots \longrightarrow \Lambda^2 T_M \longrightarrow T_M \longrightarrow \mathcal{O}(M)$$

with differential $-\imath_{dS}$, which acts on vectors by $X \mapsto -dS(X) = -X(S)$. We notice that this complex can be identified with $\text{Fun}(T^*[-1]M)$, the polyvector fields on M. In particular, we have already recovered part of our previous approach to the BV formalism, namely the odd symplectic manifold $T^*[-1]M$. This is the *classical* part of the BV formalism. The quantum part, which we shall describe presently, is intended to provide a homological approach to integral on the derived critical locus.

Remark 3.9. The choice of resolution of $\mathcal{O}(\text{graph}(dS))$ that we made above recovers $T^*[-1]M$ on the nose. When working in derived geometry, such choices of resolution are not fixed on the nose but only up to homotopy (for some appropriate notion of homotopy).

3.5.2 The de Rham Complex and Integration

The first cue for our approach to the quantum BV formalism is a reinterpretation of integration on manifolds. Consider for the moment a closed n-manifold M. Top forms on M are smooth measures, and integration defines a map

$$\int_M : \Omega^n(M) \longrightarrow \mathbb{R}$$

that descends to a map

$$\int_M : H^n_{\text{dR}}(M) \longrightarrow \mathbb{R}$$

by Stokes' Theorem.

We now describe a manoeuvre that lies at the heart of this approach to the BV formalism. Suppose we fix a top form $\omega \in \Omega^n(M)$ with non-vanishing class $[\omega] \in H^n_{\mathrm{dR}}(M)$. A simple yet compelling consequence of this choice is that the *expectation value* of a function $f \in \mathrm{Fun}(M)$ with respect to the probability measure induced by ω is given by

$$\langle f \rangle_\omega := \frac{\displaystyle\int_M f\omega}{\displaystyle\int_M \omega} = \frac{[f\omega]}{[\omega]}. \tag{26}$$

That is, the expectation value of $\langle f \rangle_\omega$ may be interpreted as a comparison of classes in cohomology, rather than as an integral proper. This simple idea[3] really lies at the heart of recent efforts to understand path integrals in quantum field theory.

Another important feature of this toy setting is that a choice of nowhere-vanishing ω gives us a map

$$\begin{aligned} \Lambda^k T_M &\xrightarrow{m_\omega} \Omega^{n-k}(M) \\ X &\longmapsto \imath_X \omega \end{aligned}$$

via contraction. The non-vanishing assumption allows us to define the coboundary operator

$$\Delta_\omega := m_\omega^{-1} \circ d \circ m_\omega, \tag{27}$$

where d is the de Rham differential. We call Δ_ω the *BV Laplacian* and we call $(\mathfrak{V}^\bullet(M), \Delta_\omega)$ the *quantum BV complex for* ω. We can now rephrase (26) as

Lemma 3.10. *For a function f on M, the cohomology class $[f]_{\mathrm{BV}} \in H^0_{\Delta_\omega}(\mathfrak{V}^\bullet(M))$ satisfies*

$$[f]_{\mathrm{BV}} = \langle f \rangle_\omega [1]_{\mathrm{BV}}.$$

To connect this to our previous discussion, we recall that $\mathfrak{V}^\bullet(M) = \mathrm{Fun}(T^*[-1]M)$ is given by functions on the shifted cotangent bundle of M. Moreover, focussing on the concrete case of $M = \mathbb{R}^n$ we write our chosen ω as

$$\omega = e^{-\frac{1}{\hbar}S(x)} dx^1 \wedge \cdots \wedge dx^n$$

(ignoring issues with overall signs for the sake of clarity). Then, denoting $\xi_i := \frac{\partial}{\partial x^i}$, an explicit computation shows that

$$\Delta_\omega = \sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi_i} + \frac{1}{\hbar}\{S, \bullet\} = \Delta + \frac{1}{\hbar}\{S, \bullet\} = \Delta - \frac{1}{\hbar}\imath_{dS},$$

with $\{\bullet, \bullet\}$ the Schouten–Nijenhuis bracket and $\Delta = \sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi_i}$ the canonical BV Laplacian (which we view as being associated to the coordinate Berezin measure on the BV manifold $T^*[-1]\mathbb{R}^n$). Note that we can also write

$$\Delta_\omega = e^{-\frac{1}{\hbar}S(x)} \Delta e^{\frac{1}{\hbar}S(x)}.$$

We now have two different complexes

$$\underbrace{(\mathrm{Fun}(T^*[-1]M), -\imath_{dS})}_{\text{classical BV}} \quad \text{and} \quad \underbrace{(\mathrm{Fun}(T^*[-1]M), \hbar\Delta - \imath_{dS})}_{\text{quantum BV}} \tag{28}$$

[3] this perspective is attributed by Gwilliam to Witten.

and the $\hbar \to 0$ limit describes the passage from the quantum to the classical theory. Conversely, we want to think of the passage from the classical to the quantum theory as a homological perturbation. Here we mean "homological perturbation" in the sense of the homological perturbation lemma, which tells us how to compute the cohomology of a complex $(V, d+\delta)$ whose differential is a "small" perturbation from that of the complex (V, d) whose cohomology is already known. While we will not make this any more precise here, the following example can be seen as evidence for the viewpoint that Feynman diagrams come from homological perturbation.

3.5.3 A Second Example: Recovering the Wick Theorem in $\mathbb{R}^n$

We will now consider an example that shows that we can recover the Wick Theorem (2) from the BV formalism in the guise of Lemma 3.10. Consider $M = \mathbb{R}^n$ equipped with the Gaussian probability measure

$$\mu_{\text{Gauss}} := \frac{(2\pi\hbar)^{\frac{n}{2}}}{\sqrt{\det A}} e^{-\frac{1}{2\hbar}(x,Ax)} dx^1 \cdots dx^n,$$

where $A = (a_{ij})$ is a symmetric positive-definite $(n \times n)$-matrix and $\hbar > 0$. As always, we are interested in computing the expectation values

$$\langle f \rangle_{\text{Gauss}} := \frac{\int_{\mathbb{R}^n} f \mu_{\text{Gauss}}}{\int_{\mathbb{R}^n} \mu_{\text{Gauss}}} = \int_{\mathbb{R}^n} f \mu_{\text{Gauss}}.$$

The first step is to rephrase this problem in terms of homological algebra. We consider the Schwartz–de Rham complex $\Omega^\bullet_{\mathcal{S}}(\mathbb{R}^n)$ of $\mathbb{R}^n$ (that is, differential forms whose coefficients are Schwartz functions). As with compactly-supported cohomology, the cohomology of $(\Omega^\bullet_{\mathcal{S}}(\mathbb{R}^n), d)$ is concentrated in degree n.

Next, for a Schwartz function $f \in \mathcal{S}$ we define $\langle f \rangle_{\text{Gauss}}$ via the expression

$$[f\mu_{\text{Gauss}}] = \langle f \rangle_{\text{Gauss}}[\mu_{\text{Gauss}}] \in H^n_{\mathcal{S}}(\mathbb{R}^n).$$

As above, μ_{Gauss} defines an isomorphism between Schwartz differential forms and Schwartz polyvector fields, so we can define the BV Laplacian as per (27). In this example, we have

$$\Delta_{\text{Gauss}} = \Delta - \frac{1}{\hbar}\sum_{i,j} a_{ij} x^i \frac{\partial}{\partial \xi_i},$$

where once again $\Delta = \sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi_i}$. An analogue of Lemma 3.10 holds in this setting, so we know that we can compute $\langle f \rangle_{\text{Gauss}}$ by comparing the class $[f]_{\text{BV}}$ to $[1]_{\text{BV}}$ in the BV complex.

Let us now consider the very simple case when $n = 1$, in which case the only non-trivial part of the BV complex is

$$\mathfrak{X}_{\mathcal{S}}(\mathbb{R}) \xrightarrow{\Delta_{\text{Gauss}}} \mathcal{S}(\mathbb{R}),$$

where

$$\Delta_{\text{Gauss}} = \frac{\partial}{\partial x}\frac{\partial}{\partial \xi} - \frac{ax}{\hbar}\frac{\partial}{\partial \xi}.$$

Taking an arbitrary element

$$F(x,\xi) = f(x)\,\xi \in \mathfrak{X}_{\mathcal{S}}(\mathbb{R}),$$

the condition $\Delta_{\text{Gauss}} F = f'(x) - \frac{ax}{\hbar} f(x) = 0$ implies $f'(x) = \frac{ax}{\hbar} f(x)$, which can only be satisfied if $f(x) \propto e^{ax^2/2\hbar}$. In this case, however, $F \notin \mathfrak{X}_{\mathcal{S}}(\mathbb{R})$ and so $H^{-1}(\text{Fun}(T^*[-1]\mathbb{R}), \Delta_{\text{Gauss}}) = 0$.

We can replace the space $\mathcal{S}$ of Schwartz functions above with the space $\mathcal{T}$ of smooth functions dominated by $e^{ax^2/2\hbar}$ and repeat the example. In this case, we get exactly the same results, except now $x^k \in \mathcal{T}$ for all k so we can compute the Gaussian moments $\langle x^k \rangle_{\text{Gauss}}$ via the BV complex. Noting that

$$\Delta_{\text{Gauss}}(x^{k-1}\xi) = (k-1)x^{k-2} - \frac{a}{\hbar} x^k,$$

we obtain

$$[x^k]_{\text{BV}} = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \left(\frac{\hbar}{a}\right)^s (2s-1)!![1]_{\text{BV}} & \text{if } k = 2s \text{ is even.} \end{cases}$$

This gives a formula for the expectation values $\langle x^k \rangle_{\text{Gauss}}$ that coincides with the Wick Theorem for $\mathbb{R}$.

Returning to the case of the Gaussian measure μ_{Gauss} on $\mathbb{R}^n$, by orthogonally diagonalising the matrix A and using what we have just done we recover the Wick Theorem for $\mathbb{R}^n$. Once we have the Wick Theorem, the regular story of Feynman diagrams follows as per usual. We have therefore recovered the usual story of Feynman diagrams *from scratch* from the BV formalism.

Concluding Remarks