C^*-algebras and K-theory

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Introduction

Idea

1. Study locally compact Hausdorff topological spaces through their algebras of continuous functions. The product on this algebra is pointwise multiplication, which is commutative.
2. Allow a suitable class of possibly noncommutative algebras: C^*-**algebras**, and view these as algebras of functions of "noncommutative spaces". This can be used to study e.g. non-Hausdorff spaces.
3. Generalise techniques from (algebraic) topology to all C^*-algebras. This works very well for K-**theory**.

Goal: Introduce C^*-algebras and their K-theory, and give examples relevant to topology and representation theory.

References

- Wegge–Olsen: K-theory and C^*-algebras (introductory);
- Blackadar: K-theory for operator algebras (comprehensive reference).

C^*-algebras

Continuous functions

From now on, let X be a locally compact Hausdorff space.

Definition

A continuous function $f : X \to \mathbb{C}$ **vanishes at infinity** if for all $\varepsilon > 0$, there is a compact subset $C \subset X$, such that for all $x \in X \setminus C$, one has $|f(x)| < \varepsilon$. (If X is compact, all functions vanish at infinity.)

Let $C_0(X)$ be the vector space of all such functions on X.

Structure on $C_0(X)$

For $f, g \in C_0(X)$ and $x \in X$, set

$$\begin{aligned} \|f\|_\infty &:= \sup_{y \in X} |f(y)|; \\ f^*(x) &:= \overline{f(x)}; \\ (fg)(x) &= f(x)g(x). \end{aligned} \tag{1}$$

Then $C_0(X)$ is a Banach space in the norm $\|\cdot\|_\infty$, and a commutative algebra over $\mathbb{C}$ with respect to the pointwise product (1).
Furthermore, we have for all $f, g \in C_0(X)$,

$$\begin{aligned} \|fg\|_\infty &\leq \|f\|_\infty \|g\|_\infty; \\ \|f^* f\|_\infty &= \|f\|_\infty^2. \end{aligned}$$

General C^*-algebras have the same structure and properties, apart from commutativity.

C^*-algebras

A C^*-**algebra** is a Banach space $(A, \|\cdot\|)$, equipped with an associative bilinear product $(a, b) \mapsto ab$ and an antilinear map $a \mapsto a^*$ whose square is the identity, such that for all $a, b \in A$, we have

$$(ab)^* = b^* a^*;$$
$$\|ab\| \leq \|a\| \, \|b\|;$$
$$\|a^* a\| = \|a\|^2.$$

A **homomorphism** of C^*-algebras is a linear homomorphism of algebras that intertwines star operations. Such homomorphisms are automatically bounded.

It follows from the C^*-algebra axioms that $\|a^*\| = \|a\|$ for all a in a C^*-algebra.

Commutative C^*-algebras

Theorem (Gelfand–Naimark)

Every commutative C^*-algebra is isomorphic to the C^*-algebra of continuous functions that vanish at infinity on a locally compact Hausdorff space. If two commutative C^*-algebras $C_0(X)$ and $C_0(Y)$ are isomorphic, then X and Y are homeomorphic.

The inverse construction to $X \mapsto A = C_0(X)$ is

Commutative C^*-algebra $A \mapsto$
Space X of nonzero homomorphisms (characters) $A \to \mathbb{C}$.

Functoriality

A **proper** continuous map f between two locally compact Hausdorff spaces X and Y induces a homomorphism of C^*-algebras

$$f^* : C_0(Y) \to C_0(X),$$

defined by pulling back functions along f. In this way, C_0 is a **contravariant functor** from the category of locally compact Hausdorff spaces, with proper continuous maps, to the category of commutative C^*-algebras.

Together with the fact that all homomorphisms between two commutative C^*-algebras $C_0(X)$ and $C_0(Y)$ are defined by pulling back along some proper continuous map, the Gelfand–Naimark theorem implies that this functor defines an **equivalence of categories**.

Dictionary

Topological spaces	C^***-algebras**
compact	unital
locally compact & Hausdorff	commutative
connected component	direct summand
Cartesian product	tensor product

Operators on Hilbert spaces

Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{B}(\mathcal{H})$ be the algebra of **bounded operators** on $\mathcal{H}$. For $a \in \mathcal{B}(\mathcal{H})$, let $\|a\|$ be the operator norm of a, and let a^* be its adjoint, defined by

$$(x, ay) = (a^*x, y)$$

for all $x, y \in \mathcal{H}$. Then $\mathcal{B}(\mathcal{H})$, equipped with these structures, is a C^*-algebra.

In fact, **all** C^*-algebras can be realised as subalgebras of an algebra of bounded operators on a Hilbert space.

Theorem (Gelfand–Naimark)

Every C^*-algebra is isomorphic to a norm-closed subalgebra of $\mathcal{B}(\mathcal{H})$ that in addition is closed under the *-operation, for some Hilbert space $\mathcal{H}$.

Example

For a suitable Borel measure on a locally compact Hausdorff space X (the counting measure always works), the representation of $C_0(X)$ in $L^2(X)$ as multiplication operators yields an embedding of $C_0(X)$ into $\mathcal{B}(L^2(X))$.

Example

The algebra $\mathcal{K}(\mathcal{H})$ of **compact operators** on a Hilbert space $\mathcal{H}$ is an ideal in $\mathcal{B}(\mathcal{H})$.

Tensor products

Let A and B be two C^*-algebras. The algebraic tensor product of A and B can be completed in various norms to give a C^*-algebra $A \otimes B$.

For some C^*-algebras A, such as commutative ones, all such completions are equal (for any B). These C^*-algebras are called **nuclear**.

$$C_0(X) \otimes C_0(Y) \cong C_0(X \times Y).$$

Group C^*-algebras

Convolution algebras

Let G be a locally compact Hausdorff topological group, equipped with a left Haar measure dg. For two functions $\varphi, \psi \in C_c(G)$, their **convolution product** $\varphi * \psi$ is defined by

$$(\varphi * \psi)(g) := \int_G \varphi(g')\psi(g'^{-1}g)dg'. \tag{2}$$

The function φ^* is defined by

$$\varphi^*(g) := \overline{\varphi(g^{-1})}\Delta(g)^{-1}, \tag{3}$$

where Δ is the modular function on G with respect to dg, defined by $d(gh) = \Delta(h)dg$ for all $h \in G$. For **unimodular** groups, Δ is the constant function 1.

The maximal group C^*-algebra

For a unitary representation $(\mathcal{H}, \pi)$ of G, and $\varphi \in C_c(G)$, we have the operator

$$\pi(\varphi) := \int_G \varphi(g)\pi(g)\, dg \quad \in \mathcal{B}(\mathcal{H}).$$

Definition

The **full/maximal C^*-algebra** $C^*(G)$ of G is the completion of the convolution algebra $C_c(G)$ in the norm

$$\|\varphi\| := \sup_{(\mathcal{H},\pi)} \|\pi(\varphi)\|_{\mathcal{B}(\mathcal{H})} \quad \leq \|\varphi\|_{L^1(G)}.$$

The supremum runs over all unitary representations $(\mathcal{H}, \pi)$ of G.

The reduced group C^*-algebra

Definition

The **reduced C^*-algebra** $C^*_r(G)$ of G is the completion of $C_c(G)$ in the norm $\|\cdot\|_r$, given by

$$\|\varphi\|_r := \|\lambda^G(\varphi)\|_{\mathcal{B}(L^2(G))}.$$

Here $\lambda^G : G \to \mathrm{U}(L^2(G))$ is the **left regular representation**

$$(\lambda^G(g)\varphi)(g') = \varphi(g^{-1}g').$$

Note that $\lambda^G(\varphi)\psi = \varphi * \psi$ for all $\varphi \in C_c(G)$ and $\psi \in L^2(G)$.

The group G is called **amenable** if $C^*_r(G) = C^*(G)$. I.e. all irreducible unitary representations occur in $(L^2(G), \lambda^G)$.
Examples: G abelian or compact.

The unitary dual

Consider the **unitary dual** $\hat{G}$ and the **tempered unitary dual** $\hat{G}_{\text{temp}}$ of G:

- $\hat{G} := \{\text{unitary irreducible representations of } G\}$;
- $\hat{G}_{\text{temp}} := \{(\mathcal{H}, \pi) \in \hat{G}; \mathcal{H} \hookrightarrow L^2(G)\}$.

There is a natural topology on $\hat{G}$, which is not Hausdorff in general.

Idea:

- $C^*(G) =$ "$\{\text{noncommutative functions on } \hat{G}\}$";
- $C^*_r(G) =$ "$\{\text{noncommutative functions on } \hat{G}_{\text{temp}}\}$".

This is true for **abelian** groups G:

- $\hat{G}_{\text{temp}} = \hat{G}$, and
- $C^*(G) = C^*_r(G) = C_0(\hat{G})$,

via the Fourier transform.

The C^*-algebra of a compact group

Let K be a compact Lie group, and consider the direct sum

$$\bigoplus_{(V,\pi)\in\hat{K}} \mathcal{B}(V) := \{(a_\pi)_{(V,\pi)\in\hat{K}}; a_\pi \in \mathcal{B}(V), \lim_{\pi\to\infty} \|a_\pi\|_{\mathcal{B}(V)} = 0\}.$$

I.e. C_0-sections of the bundle

$$\big(\mathcal{B}(V)\big)_{(V,\pi)\in\hat{K}} \to \hat{K}.$$

Equipped with the norm

$$\|(a_\pi)_{\pi\in\hat{K}}\| := \sup_{(V,\pi)\in\hat{K}} \|a_\pi\|_{\mathcal{B}(V)},$$

this becomes a C^*-algebra.

The C^*-algebra of a compact group

Proposition

There is an isomorphism of C^*-algebras

$$C^*_r(K) = C^*(K) \cong \bigoplus_{(V,\pi)\in\hat{K}} \mathcal{B}(V).$$

The right hand side is very similar to $C_0(\hat{K})$, up to replacing $\mathbb{C}$ by matrix algebras $\mathcal{B}(V)$.

Proof

Consider the Hilbert space $\hat{L}^2(K) :=$

$$\Big\{a = (a_\pi)_{(V,\pi)\in\hat{K}};\, a_\pi \in \mathcal{B}(V), (a,a) := \sum_{\pi\in\hat{K}} \mathrm{tr}(a_\pi^* a_\pi) < \infty\Big\}.$$

Peter–Weyl theorem: the Plancherel transform $P : L^2(K) \to \hat{L}^2(K)$,

$$(Pf)_{(V,\pi)} = \sqrt{\dim V}\,\pi(f)$$

for $f \in L^2(K)$ and $\pi \in \hat{K}$, is a unitary isomorphism.
The map $\varphi : C^*(K) \to \mathcal{B}(\hat{L}^2(K))$,

$$\varphi(f) = P\pi(f)P^{-1},$$

for $f \in C(K)$, is isomorphism of C^*-algebras onto its image, which is $\bigoplus_{(V,\pi)\in\hat{K}} \mathcal{B}(V)$. □

Roe algebras

The natural C^* algebras used in **coarse geometry** are **Roe algebras**.

Let

- (X, d) be a metric space in which all closed balls are compact;
- G be a locally compact group acting properly and isometrically on X;
- dx be a G-invariant Borel measure on X for which every nonempty open set has positive volume;
- $\mathcal{H}$ be any infinite-dimensional separable Hilbert space, such as $l^2(\mathbb{N})$.

Definition

An operator $T \in \mathcal{B}(L^2(X, \mathcal{H}; dx))$ has **finite propagation** if there is an $R > 0$, such that for all $f_1, f_2 \in C_0(X)$ with supports at least a distance R apart, $f_1 T f_2 = 0$.

Roe algebras

Definition

The **reduced equivariant Roe algebra** of X is the closure $C^*_G(X)$ in $\mathcal{B}(L^2(X, \mathcal{H}; dx))$ of

$$\{T \in \mathcal{B}(L^2(X, \mathcal{H}; dx)); fT \text{ and } Tf \text{ compact}; \\ T \text{ has finite propagation}; \\ T \text{ is } G\text{-equivariant}\}.$$

(**Idea:** G-invariant operators with smooth kernels supported near the diagonal. E.g. finitely propagating parametrices of G-equivariant elliptic operators on X.)

There is also a **maximal equivariant Roe algebra**, completed in a norm similar to the maximal group C^*algebra norm.

Roe algebras and group C^*-algebras

Roe algebras generalise group C^*-algebras, in the sense that

$$C^*_G(|G|) \cong C^*(G) \otimes \mathcal{K}(\mathcal{H}).$$

- $|G|$ is the group G, considered as a metric space with respect to a left invariant metric for which all closed balls are compact.
- We will see that tensoring with $\mathcal{K}(\mathcal{H})$ does not change the K-theory of a C^*-algebra.
- This holds for reduced and maximal Roe algebras and group C^*-algebras.

K-theory

Topological K-theory

We first consider a **compact** Hausdorff space X.

Definition

The **(topological)** K**-theory** of X is the abelian group $K^0(X)$ whose generators are isomorphism classes $[E]$ of (complex) vector bundles $E \to X$, subject to the relation

$$[E] + [F] = [E \oplus F]$$

for all vector bundles E and F over X.

Note that a general element of $K^0(X)$ is a formal difference $[E] - [F]$ of isomorphism classes of vector bundles.

Functoriality

A continuous map $f : X \to Y$ between compact Hausdorff spaces induces a map $f^* : K^0(Y) \to K^0(X)$, defined via the **pullback** of vector bundles along f.

This turns K^0 into a **contravariant functor** from the category of compact Hausdorff spaces to the category of abelian groups.

K-theory vs. cohomology

Theorem

For compact X, there is an isomorphism **(Chern character)**

$$\chi^0 : K^0(X) \otimes \mathbb{Q} \xrightarrow{\cong} \bigoplus_{n \text{ even}} H^n(X; \mathbb{Q}).$$

Here $H^n(X; \mathbb{Q})$ is the nth (e.g. Čech) cohomology group of X with rational coeficients.

(There is an analogous isomorphism for odd K-theory, which will be defined later.)

Noncompact spaces

Let $X^+ = X \cup \{\infty\}$ be the one-point compactification of a locally compact Hausdorff space X. Let

$$i : \{\infty\} \hookrightarrow X^+$$

be the inclusion map of the point at infinity. Consider the functorially induced map

$$i^* : K^0(X^+) \to K^0(\{\infty\}).$$

Vector bundles over the one-point space $\{\infty\}$ are just finite-dimensional vector spaces. Therefore $K^0(\{\infty\}) \cong \mathbb{Z}$, via the dimension map.

Definition

The K-theory of the locally compact Hausdorff space X is the kernel of the map i^*. It is denoted by $K^0(X)$.

Functoriality

A continuous map between locally compact Hausdorff spaces

$$f : X \to Y$$

induces a map

$$f^* : K_0(Y) \to K_0(X)$$

if it extends continuously to a map between the one-point compactifications of X and Y.

This means that f should be **proper** ($f^{-1}(C)$ is compact if $C \subset Y$ is compact). Hence topological K-theory is a **contravariant functor** from the category of locally compact Hausdorff spaces, with proper continuous maps, to the category of abelian groups.

Vector bundles and $C(X)$-modules

Suppose that X is compact, so that $C_0(X) = C(X)$.
If $E \to X$ is a vector bundle, then the space $\Gamma(E)$ of its continuous sections has the natural structure of a $C(X)$-**module**, given by pointwise multiplication.

- $\Gamma(E) \cong \Gamma(F)$ as $C(X)$-modules iff $E \cong F$.
- $\Gamma(E \oplus F) \cong \Gamma(E) \oplus \Gamma(F)$.
- For any vector bundle $E \to X$, there is a vector bundle $F \to X$ such that $E \oplus F \cong X \times \mathbb{C}^n$.

Then

$$\Gamma(E) \oplus \Gamma(F) \cong \Gamma(E \oplus F) = \Gamma(X \times \mathbb{C}^n) \cong C(X)^n.$$

Finitely generated projective modules

More generally, a module $\mathfrak{M}$ over a C^*-algebra (or ring) A is called **finitely generated** and **projective** if there exists an A-module $\mathfrak{N}$ such that $\mathfrak{M} \oplus \mathfrak{N}$ is a finitely generated free A-module, i.e. of the form A^n for some $n \in \mathbb{N}$.

Any finitely generated projective $C(X)$-module is isomorphic to the module $\Gamma(E)$, for some vector bundle $E \to X$.

Theorem (Serre–Swan)

The K-theory of the compact Hausdorff space X is the abelian group whose generators are isomorphism classes $[\mathfrak{M}]$ of finitely generated projective $C(X)$-modules, subject to the relation

$$[\mathfrak{M}] + [\mathfrak{N}] = [\mathfrak{M} \oplus \mathfrak{N}]$$

for all finitely generated projective modules $\mathfrak{M}$ and $\mathfrak{N}$ over $C(X)$.

K-theory of unital C^*-algebras

Let A be a C^*-algebra with a unit.

Definition

The K-**theory** of A is the abelian group whose generators are isomorphism classes $[\mathfrak{M}]$ of finitely generated projective A-modules, subject to the relation

$$[\mathfrak{M}] + [\mathfrak{N}] = [\mathfrak{M} \oplus \mathfrak{N}]$$

for all finitely generated projective modules $\mathfrak{M}$ and $\mathfrak{N}$ over A.

Functoriality

Let

$$f : A \to B$$

be a unital $*$-homomorphisms between unital C^*-algebras, and $\mathfrak{M}$ a finitely generated projective (right) A-module. Form the f.g. projective B-module

$$\mathfrak{M} \otimes_f B := \mathfrak{M} \otimes_{\mathbb{C}} B/(m \cdot a) \otimes b \sim m \otimes (f(a)b).$$

for all $m \in \mathfrak{M}$, $a \in A$ and $b \in B$.
This induces a map

$$f_* : K_0(A) \to K_0(B).$$

This makes the K-theory of unital C^*-algebras a **covariant functor**. (Hence the **subscript** 0.)

Pullbacks of vector bundles

Functoriality of C^*-algebraic K-theory generalizes functoriality of topological K-theory.

Lemma

Let X and Y be compact Hausdorff spaces, let $f : X \to Y$ be a continuous map, and let $E \to Y$ be a vector bundle. Consider the homomorphism of C^*-algebras $f^* : C(Y) \to C(X)$ defined by pulling back functions along f. There is an isomorphism

$$\Gamma(X, f^*E) \cong \Gamma(Y, E) \otimes_{f^*} C(X).$$

Compactification and unitisation

The extension of K-theory to non-unital C^*-algebras is analogous to the extension of topological K-theory to noncompact spaces.
If X is a locally compact Hausdorff space, then

$$C_0(X)^+ := C_0(X) \oplus \mathbb{C} \cong C(X^+).$$

The multiplication, star operation and the norm on $C_0(X)^+$ are defined by

$$\begin{aligned} (f+z)(g+w) &:= fg + zg + wf + zw; \\ (f+z)^* &:= f^* + \bar{z}; \\ \|f+z\| &:= \sup_{x \in X} |f(x) + z| = \|f+z\|_{\mathcal{B}(C_0(X))}, \end{aligned}$$

for $f, g \in C_0(X)$ and $z, w \in \mathbb{C}$. The resulting C^*-algebra $C_0(X)^+$ is called the **unitisation** of $C_0(X)$.

Compactification and unitisation

The inclusion map $i : \{\infty\} \hookrightarrow X^+$ induces the map

$$i^* : C_0(X^+) \cong C_0(X)^+ \to \mathbb{C} \tag{4}$$

given by the natural projection onto the term $\mathbb{C}$. Then we have

Proposition

The topological K-theory of X is the kernel of the map

$$i^* : K_0(C(X^+)) \to K_0(\mathbb{C}) \cong \mathbb{Z}$$

induced by (4).

Unitisations

For a general C^*-algebra, we proceed as follows.

Definition

Let A be a C^*-algebra. Its **unitisation** A^+ is defined as the algebra $A^+ := A \oplus \mathbb{C}$, with multiplication, star operation and norm given by

$$\begin{aligned}(a+z)(b+w) &:= ab + zb + wa + zw;\\ (a+z)^* &:= a^* + \bar{z};\\ \|a+z\|_{A^+} &:= \|a+z\|_{\mathcal{B}(A)},\end{aligned}$$

for $a, b \in A$ and $z, w \in \mathbb{C}$. Here $\|a+z\|_{\mathcal{B}(A)}$ is the norm of $a+z$ as a bounded operator on the Banach space A, given by left multiplication.

K-theory of nonunital C^*-algebras

For a C^*-algebra A, consider the map

$$\begin{aligned} i^* : A^+ &\to \mathbb{C}, \\ a + z &\mapsto z. \end{aligned}$$

We denote the induced map on K-theory by

$$i^* : K_0(A^+) \to K_0(\mathbb{C}) \cong \mathbb{Z}.$$

Definition

The K-**theory** of A is the kernel of the map i^*. It is denoted by $K_0(A)$.

Hence for all locally compact Hausdorff spaces, we have $K^0(X) = K_0(C_0(X))$.

K-theory via projections

The K-theory of a unital C^*-algebra A is often defined in terms of **projections** in the infinite matrix algebra

$$M_\infty(A) := \varinjlim M_n(A),$$

i.e. elements p such that $p^2 = p = p^*$. These correspond to f.g. projective A-modules via $p \mapsto p\big(A^n\big)$, for p a projection in $M_n(A)$.

The functoriality of K-theory is then induced by

$$f(p)_{ij} = f(p_{ij}) \quad \in B,$$

if $f : A \to B$ is a homomorphism of C^*-algebras and $p \in M_\infty(A)$ is a projection.

Unital vs. nonunital algebras

In the projection picture another reason why K-theory for non-unital C^*-algebras has to be defined separately becomes apparent.

Indeed, if X is a connected, locally compact but not compact Hausdorff space, then there are no nonzero projections in $M_\infty(C_0(X))$, because the trace of such a projection is a constant function on X.

Higher K-groups

For any integer n, and any C^*-algebra A, one has the K-theory group

$$K_n(A) := K_0\big(A \otimes C_0(\mathbb{R}^n)\big).$$

Bott periodicity is the statement that $K_{n+2}(A) \cong K_n(A)$ for all such n and A (naturally in A). Therefore, it is enough to consider the K-theory groups $K_0(A)$ and $K_1(A)$.

(There is also a direct characterisation of $K_1(A)$, in terms of the group of connected components of the invertible matrices over A.)

Examples

A	$K_0(A)$	$K_1(A)$
$\mathbb{C}$	$\mathbb{Z}$	0
$M_n(\mathbb{C})$	$\mathbb{Z}$	0
$\mathcal{K}(\mathcal{H})$	$\mathbb{Z}$	0
$\mathcal{B}(\mathcal{H})$	0	0
$\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$	0	$\mathbb{Z}$
$C_0(\mathbb{R}^{2n})$	$\mathbb{Z}$	0
$C_0(\mathbb{R}^{2n+1})$	0	$\mathbb{Z}$
$C(S^{2n})$	$\mathbb{Z}^2$	0
$C(S^{2n+1})$	$\mathbb{Z}$	$\mathbb{Z}$
$C(\mathbb{T}^n)$	$\mathbb{Z}^{2^{n-1}}$	$\mathbb{Z}^{2^{n-1}}$

- $\mathcal{H}$ separable, infinite-dimensional;
- **Stability:** $K_j(A \otimes \mathcal{K}(\mathcal{H})) = K_j(A)$.

Half-exactness

Theorem
If

$$0 \to J \to A \to A/J \to 0$$

is an exact sequence of C^*-algebras, then the sequences

$$K_j(J) \to K_j(A) \to K_j(A/J)$$

induced on K-theory, are exact in the middle.

This can be extended to a long exact cohomology sequence, which is periodic because of Bott periodicity.

The six term exact sequence

Let

$$0 \to J \to A \to A/J \to 0$$

be an exact sequence of C^*-algebras.

Theorem

There is aboundary map

$$\partial : K_{j+1}(A/J) \to K_j(J),$$

such that the following diagram is exact:

$$\begin{array}{ccccc}
K_0(J) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/J) \cong K_2(A/J) \\
\partial \uparrow & & & & \downarrow \partial \\
K_1(A/J) & \longleftarrow & K_1(A) & \longleftarrow & K_1(J).
\end{array}$$

K-theory of group C^*-algebras

The K-theory of the C^*-algebra of an abelian group

If G is an **abelian** group, then we saw that

$$C^*(G) = C^*_r(G) = C_0(\hat{G}).$$

So

$$K_j(C^*(G)) = K_j(C^*_r(G)) = K_j(C_0(\hat{G})) = K^j(\hat{G}).$$

In general, the rough idea is that

$$\text{“}K_j(C^*(G)) = K^j(\hat{G})\text{”};$$
$$\text{“}K_j(C^*_r(G)) = K^j(\hat{G}_{\text{temp}})\text{”}.$$

The representation ring

Let K be a compact Lie group.

Definition

The **representation ring** $R(K)$ of K is the abelian group generated by equivalence classes of finite-dimensional representations of K, subject to the relation

$$[V] + [W] = [V \oplus W],$$

for irreducible representations (V, π) and (W, ρ) of K.

Explicitly,

$$\begin{aligned} R(K) &= \{[V] - [W]; (V, \pi), (W, \rho) \text{ fin. dim. reps. of } K\}; \\ &= \Big\{ \sum_{(V,\pi) \in \hat{K}} m_\pi [V]; m_\pi \in \mathbb{Z}, \text{ finitely many nonzero} \Big\}. \end{aligned}$$

The K-theory of the C^*-algebra of a compact group

For compact groups K, the K-theory group $K_0(C^*(K))$ is isomorphic to (the abelian group underlying) the representation ring $R(K)$, while $K_1(C^*(K)) = 0$.

Indeed, let (V, π) be a finite-dimensional representation of K, $f \in C(K)$, $v \in V$. Set

$$f \cdot v := \pi(f)v = \int_K f(k)\pi(k)v\, dk, \tag{5}$$

for a Haar measure dk on K. This extends to a f.g. projective $C^*(K)$-module structure on V.

Proposition

This procedure induces an isomorphism of abelian groups

$$R(K) \cong K_0(C^*(K)). \tag{6}$$

Proof

The proof is based on the facts that

- $C^*(K) \cong \bigoplus_{(V,\pi)\in\hat{K}} \mathcal{B}(V)$;
- K-theory preserves finite direct sums;
- K-theory preserves inductive limits;
- $K_0(\mathcal{B}(V)) = \mathbb{Z}\cdot[V]$.

Therefore,

$$K_0(C^*(K)) = \bigoplus_{(V,\pi)\in\hat{K}} K_0(\mathcal{B}(V)) = \bigoplus_{(V,\pi)\in\hat{K}} \mathbb{Z}\cdot[V] = R(K).$$

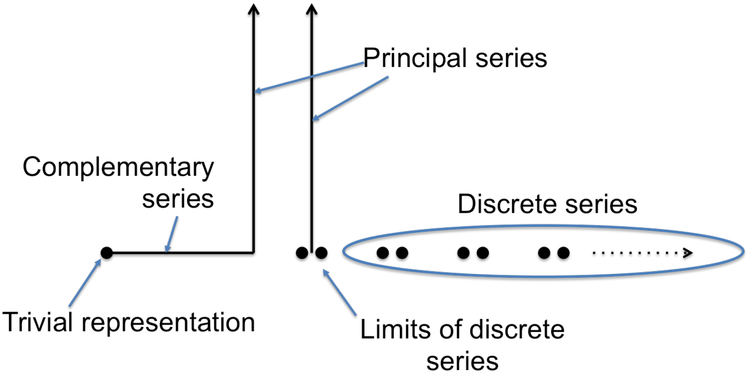
$\square$

(Recall that for compact groups, the full and reduced C^*-algebras coincide.)

The unitary dual of a noncompact, nonabelian group

Unitary irreducible representations of

$$SL_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}); \det A = 1\} :$$



$K_0(C^*(G))$ for semisimple Lie groups

Let G be a semisimple Lie group with **discrete series** representations π, i.e. all matrix coefficients $g \mapsto (x, \pi(g) \cdot y)$ are in $L^2(G)$.

Let $(\mathcal{H}, \pi)$ be a discrete series representation of G. Fix $x \in \mathcal{H}$ of norm 1, and define $f_\pi \in L^2(G)$ by

$$f_\pi(g) = (x, g \cdot x)_{\mathcal{H}}.$$

Set

$$d_\pi := \|f_\pi\|^{-1}_{L^2(G)},$$

the **formal degree** of π. Then $d_\pi f_\pi \in C^*_r(G)$ is a projection.

$$[\pi] := [d_\pi f_\pi] \in K_0(C^*_r(G))$$

embeds the discrete series into $K_0(C^*_r(G))$. ($K_1(C^*_r(G)) = 0$.)

Generalising $K_*(C^*(G))$

K-theory of the equivariant **Roe algebras** defined before generalises K-theory of group C^*-algebras.

Let (X, d) be a metric space, on which a locally compact group G acts properly, freely and isometrically, such that X/G is **compact**. Then

$$K_j(C^*_G(X)) = K_j(C^*(G)),$$

with $C^*_G(X)$ the equivariant Roe algebra of X. This equality holds for both reduced and maximal Roe algebras and group C^*-algebras.