The Hitchin fibration and real forms through spectral data

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The plan

1. Higgs bundles.
   - Classical Higgs bundles
   - G^c-Higgs bundles.
2. The Hitchin fibration
   - Spectral data approach.
   - G-Higgs bundles as fixed points.
3. Spectral data approach for real forms
   - Discrete fixed point set.
   - Pos. dim. fixed point set.
   - No fixed point set.
4. Applications
   - Connected components.
   - Topological invariants.

Based on:
arXiv:1301.1981
arXiv:1111.2550

Spectral data also used in arXiv:1305.4638 with D. Baraglia.

CLASSICAL HIGGS BUNDLES

Σ compact Riemann surface of genus $g > 2$,
canonical bundle $K = T^*\Sigma$.

Definition

A *Higgs bundle* on a compact Riemann surface Σ of genus $g > 1$, is a pair (E, Φ) for E a holomorphic vector bundle on Σ and Φ a section in $H^0(\Sigma, \text{End}(E) \otimes K)$. See example.

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The *slope* of a vector bundle F be $\mu := \deg(F)/\mathrm{rk}(F)$. A Higgs bundle (E, Φ) is

- *stable* if for each Φ-invariant subbundle F one has $\mu(F) < \mu(E)$;
- *semi* $-$ *stable* if for each Φ-invariant subbundle F one has $\mu(F) \leq \mu(E)$;
- *polystable* if $(E, \Phi) = (E_1, \Phi_1) \oplus (E_2, \Phi_2) \oplus \ldots (E_r, \Phi_r)$, where (E_i, Φ_i) is stable with $\mu(E_i) = \mu(E)$ for all i.

See example.

G^c-Higgs bundles

G^c be a complex semisimple Lie group.

Definition

A G^c-Higgs bundle is a pair (P, Φ) where

- P is a principal G^c-bundle over Σ,
- Higgs field Φ is a holomorphic section of the vector bundle $\mathrm{Ad}P \otimes_{\mathbb{C}} K$,

for ad$P$ the vector bundle associated to the adjoint representation.

- Can extend stability notions. Then, call $\mathcal{M}_{G^c}$ the moduli space of S-equivalence classes of semi-stable G^c-Higgs bundles.

G^c-Higgs bundles: Examples

- Classical Higgs bundles are given by $GL(n,\mathbb{C})$-Higgs bundles.

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G^c-HIGGS BUNDLES: EXAMPLES

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- For $G^c \subset GL(n,\mathbb{C})$ one has classical Higgs bundles + extra conditions:

- $SL(n,\mathbb{C})$

$$(E,\Phi) \text{ for } \begin{cases} E \text{ rk } n \text{ vector bundle s.t. } \Lambda^n E \cong \mathcal{O} \\ \Phi : E \to E \otimes K \quad \text{s.t. } \mathrm{Tr}(\Phi) = 0 \end{cases}$$

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- $Sp(2n,\mathbb{C})$

$$(E,\Phi) \ \text{ for } \left\{ \begin{array}{c} E \text{ rk } 2n \text{ symplectic vector bundle} \\ \text{symplectic form } \omega \text{ on } E \\ \Phi : E \to E \otimes K \quad \text{s.t. } \omega(\Phi v, w) = -\omega(v, \Phi w) \end{array} \right.$$

THE HITCHIN FIBRATION

Let d_i, for $i = 1, \ldots, k$, be the degrees of the basic invariant polynomials p_i on the Lie algebra $\mathfrak{g}^c$ of G^c.

Definition

The Hitchin fibration is

$$\begin{array}{rcl} h \ : \ \mathcal{M}_{G^c} & \longrightarrow & \mathcal{A}_{G^c} := \bigoplus_{i=1}^{k} H^0(\Sigma, K^{d_i}), \\ (E, \Phi) & \mapsto & (p_1(\Phi), \ldots, p_k(\Phi)). \end{array}$$

- h is a proper map and $\dim \mathcal{A}_{G^c} = \dim \mathcal{M}_{G^c}/2$.
- The Hitchin map makes the Higgs bundles moduli space into an integrable system.
- For most classical groups (not $SO(2n, \mathbb{C})$), we take polynomials on $\mathrm{Tr}(\Phi^i)$ for a basis of invariant polynomials.

See examples.

N.Hitchin '87

Spectral data approach

The idea

$$\mathcal{M}_{G^c} \longrightarrow \mathcal{A}_{G^c} = \bigoplus_{i=1}^{k} H^0(\Sigma, K^{d_i})$$

$$(E, \Phi) \longmapsto \mathrm{char}(\Phi) \rightsquigarrow a = (a_1, \ldots, a_k)$$

$\rightsquigarrow$ S a d_k−fold cover of Σ

S a d_k−fold cover of Σ $\longrightarrow$ Spectral data

S a d_k−fold cover of Σ $\rightsquigarrow$ data on S

data on S $\longrightarrow$ Spectral data

SPECTRAL DATA APPROACH: $GL(n,\mathbb{C})$

$$h \; : \; (E,\Phi) \mapsto \mathrm{char}(\Phi) \in \bigoplus_{i=1}^{n} H^0(\Sigma, K^i) = \mathcal{A}_{GL(n,\mathbb{C})}$$

$$\mathrm{char}(\Phi) = \eta^n + a_1\eta^{n-1} + a_2\eta^{n-2} + \ldots + a_{n-1}\eta + a_n$$

So the fibration looks as follows...

N.Hitchin ‘87, ‘07

Spectral data approach: $GL(n,\mathbb{C})$

The construction

Starting with (S,M) we get a stable Higgs bundle (E,Φ) for

- The rank n vector bundle $E=\rho_*M$;
- The Higgs field Φ induced by

$$H^0(\rho^{-1}(\mathcal{U}),M)\xrightarrow{\eta}H^0(\rho^{-1}(\mathcal{U}),M\otimes\rho^*K)$$

for an open $\mathcal{U}\subset\Sigma$ by definition of direct image, gives

$$H^0(\mathcal{U},\rho_*M)\longrightarrow H^0(\mathcal{U},\rho_*M\otimes K)$$

Pushes down to $\Phi:E\to E\otimes K$.

Starting with a stable (E, Φ) we get the spectral data (S, M) for

- The smooth spectral curve S defined by $\det(\eta - \rho^*\Phi) = 0$;
- For the line bundle $U := \operatorname{coker}(\eta - \rho^*\Phi)$, one has $\rho_* U = E$.

The generic fibre of the Hitchin fibration is isomorphic to the Jacobian of the spectral curve S

Spectral data approach: $SL(n, \mathbb{C})$

As classical Higgs bundles + extra data.

$$\Lambda^n E = \mathcal{O} \leftrightsquigarrow \Lambda^n \rho_* M \cong \mathcal{O}$$

From [Beauville-Narasimhan-Ramanan, 1989],

$$\Lambda^n \rho_* M \cong \mathrm{Nm}(M) \otimes K^{-n(n-1)/2}.$$

$$\begin{array}{rcl} \mathrm{Nm} : \mathrm{Pic}(S) & \to & \mathrm{Pic}(\Sigma) \\ \sum n_i p_i & \mapsto & \sum n_i \rho(p_i) \end{array}$$

Then $\Lambda^n \rho_* M = \mathcal{O}$ if and only if $\mathrm{Nm}(M) \cong K^{n(n-1)/2}$, or equivalently

$$M \otimes \rho^* K^{-(n-1)/2} \in \mathrm{Ker}(\mathrm{Nm}) =: \mathrm{Prym}(S, \Sigma).$$

The generic fibre of the Hitchin fibration is biholomorphically equivalent to the Prym variety $\mathrm{Prym}(S, \Sigma)$.

G-HIGGS BUNDLES

SET UP

- G a real reductive Lie group;
- $\mathfrak{g}^{\mathbb{C}}$ complexified Lie algebra of G;
- $\mathfrak{m}^{\mathbb{C}}$ such that

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$$

- $H \subset G$ the maximal compact subgroup;
- $\mathfrak{h}^{\mathbb{C}}$ complexified Lie algebra of H;
- $Ad|_{H^{\mathbb{C}}} : H^{\mathbb{C}} \to GL(\mathfrak{m}^{\mathbb{C}})$ is the isotropy representation.

Definition

A principal G-Higgs bundle is a pair (P, Φ) where

- *P is a holomorphic principal $H^{\mathbb{C}}$-bundle;*
- *Φ is a holomorphic section of $(P \times_{Ad} \mathfrak{m}^{\mathbb{C}}) \otimes K$.*

See example.

G-Higgs bundles as fixed points

in the fibres of the G^c Hitchin fibration

- G a real form of G^c fixed by the anti-holomorphic involution τ
- ρ the compact real form of G^c.
- $\sigma = \rho\tau$ a holomorphic involution.

$$\Theta : (E, \Phi) \mapsto (\sigma(E), -\sigma(\Phi))$$

N.Hitchin ‘87, ‘92

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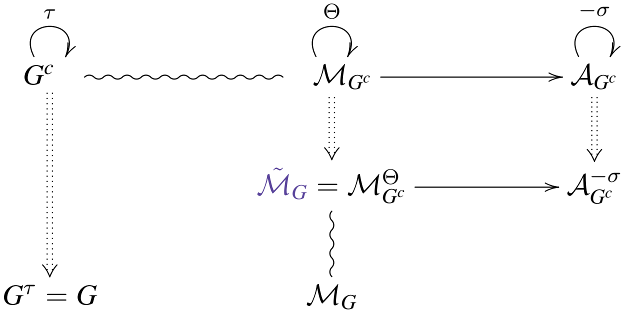


See Example.

N.Hitchin '87, García-Prada – Gothen – Mundet '09

Spectral data approach I

Discrete intersection of $\mathcal{M}^{\Theta}_{G^c}$ with the smooth fibres

G-Higgs bundles for split real forms ($G = SL(n, \mathbb{R}),\ Sp(2n, \mathbb{R}), \ldots$)

Theorem (*thesis*)

The intersection of $\mathcal{M}^{\Theta}_{G^c}$ with the smooth fibres of the Hitchin fibration

$$h : \mathcal{M}_{G^c} \to \mathcal{A}_{G^c}.$$

is the space of elements of order 2 over the regular locus of $\mathcal{A}_{G^c}$.

So we can study the fibration through the monodromy action...
$G = SL(2, \mathbb{R})$ case (*thesis*)

What happens for other groups?

Spectral data approach II

Positive dimensional intersection of $\mathcal{M}_{G^c}^{\Theta}$ with the smooth fibres

What kind of curve does char(Φ) define for a G-Higgs field Φ?

Generically, a smooth curve for $G = U(p,p),\ SU(p,p)$...

Definition

A $U(p,p)$-Higgs bundle over Σ is a pair (E,Φ) where $E = V \oplus W$ for V, W rank p vector bundles over Σ, and Φ the Higgs field given by

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix},$$

for $\beta : W \to V \otimes K$ and $\gamma : V \to W \otimes K$. When $\Lambda^p V \cong \Lambda^p W^*$, one has an $SU(p,p)$-Higgs bundle.

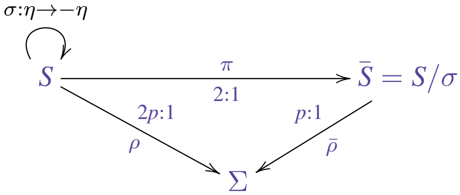
$$\det(x - \Phi) = x^{2p} + a_1 x^{2p-2} + \ldots + a_{p-1}x^2 + a_p$$

Spectral data approach II

Towards the spectral data for $U(p,p)$-Higgs bundles

- Smooth S given by $\eta^{2p} + a_1\eta^{2p-2} + \ldots + a_{p-1}\eta^2 + a_p = 0$;
- Smooth $\bar{S}$ given by $\xi^p + a_1\xi^{p-1} + \ldots + a_{p-1}\xi + a_p = 0$ for $\xi = \eta^2$;

for η tautological section of ρ^*K and $a_i \in H^0(\Sigma, K^{2i})$.



Can be adapted to study $SU(p,p)$-Higgs bundles...

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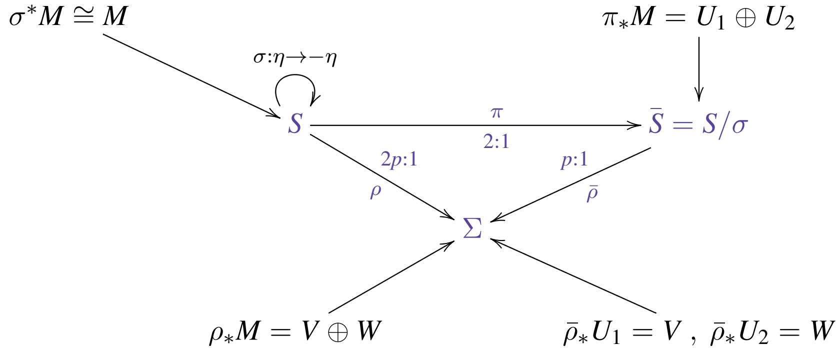
for η tautological section of ρ^*K and $a_i \in H^0(\Sigma, K^{2i})$.

- $M := \text{coker}(\rho^*\Phi - \eta)$ line bundle on S.



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Spectral data approach II

Spectral data for $U(p,p)$-Higgs bundles (*thesis*)

There is a one to one correspondence between $U(p,p)$-Higgs bundles $(V \oplus W, \Phi)$ on Σ with $\deg V > \deg W$ and non-singular spectral curve, and triples $(\bar{S}, U_1, D)$ where

- $\bar{\rho} : \bar{S} \to \Sigma$ is an irreducible non-singular p-cover of Σ in the total space of K with equation

$$\xi^p + a_1 \xi^{p-1} + \ldots + a_{p-1}\xi + a_p = 0,$$

  for $a_i \in H^0(\Sigma, K^{2i})$, and ξ the tautological section of $\bar{\rho}^* K^2$.
- U_1 is a line bundle on $\bar{S}$ whose degree is

$$\deg U_1 = \deg V + (2p^2 - 2p)(g-1)$$

- D is a positive subdivisor of the divisor of a_p of degree

$$\tilde{m} = \deg W - \deg V + 2p(g-1).$$

SPECTRAL DATA APPROACH II

SPECTRAL DATA FOR $U(p,p)$-HIGGS BUNDLES: THE INVARIANTS

Since $\sigma^* M \cong M$ then

$$H^0(\rho^{-1}(\mathcal{U}), M) = H^0(\rho^{-1}(\mathcal{U}), M)^+ \oplus H^0(\rho^{-1}(\mathcal{U}), M)^-$$

$$\begin{aligned} h^+ := \dim H^0(\rho^{-1}(\mathcal{U}), M)^+ &= \dim H^0(\mathcal{U}, V), \\ h^- := \dim H^0(\rho^{-1}(\mathcal{U}), M)^- &= \dim H^0(\mathcal{U}, W). \end{aligned}$$

Use the L_σ Lefschetz number [Atiyah-Bott 1968] associated to the involution σ on S

$$\begin{aligned} L_\sigma &= \sum (-1)^q \mathrm{trace}\, \sigma|_{H^{0,q}(M)} = \mathrm{trace}\, \sigma|_{H^0(M)} = h^+ - h^- \\ L_\sigma &= \frac{(-\tilde{m}) + (4p(g-1) - \tilde{m})}{2} = 2p(g-1) - \tilde{m}. \end{aligned}$$

$$\begin{aligned} \deg U_1 &= v + (2p^2 - 2p)(g-1) = \frac{\deg M}{2} - \frac{\tilde{m}}{2}, \\ \deg U_2 &= w + (2p^2 - 2p)(g-1) = \frac{\deg M}{2} + \frac{\tilde{m}}{2} - 2p(g-1). \end{aligned}$$

SPECTRAL DATA APPROACH III

NO INTERSECTION OF $\mathcal{M}_{G^c}^{\Theta}$ WITH THE SMOOTH FIBRES

What kind of curve does char(Φ) define for a G-Higgs field Φ?

Generically, a reducible curve for $G = Sp(2p, 2p),\ SU(p, q), \ldots\ (p \neq q)$

Definition

An $Sp(2p, 2p)$-Higgs bundle is a pair $(V \oplus W, \Phi)$ for V and W rank $2p$ symplectic vector bundles, and the Higgs field

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \text{ for } \begin{cases} \beta : W \to V \otimes K \\ \gamma : V \to W \otimes K \end{cases} \text{ and } \boxed{\beta = -\gamma^{\mathrm{T}},}$$

for γ^{T} the symplectic transpose of γ.

$$\det(x - \Phi) = (x^{2p} + a_1 x^{2p-2} + \ldots + a_{p-1} x^2 + a_p)^2$$

Note this is the case of $SU^*(2p)$ and $SO^*(2p)$, current work w/ N. Hitchin to appear soon.

Spectral data approach III

Towards the spectral data for $Sp(2p,2p)$-Higgs bundles

- ∘ Smooth S given by $\eta^{2p} + a_1\eta^{2p-2} + \ldots + a_{p-1}\eta^2 + a_p = 0$;
- ∘ Smooth $\bar{S}$ given by $\xi^p + a_1\xi^{p-1} + \ldots + a_{p-1}\xi + a_p = 0$ for $\xi = \eta^2$;

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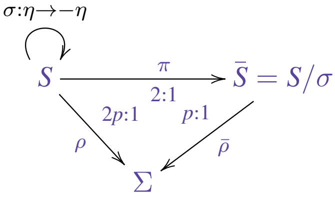

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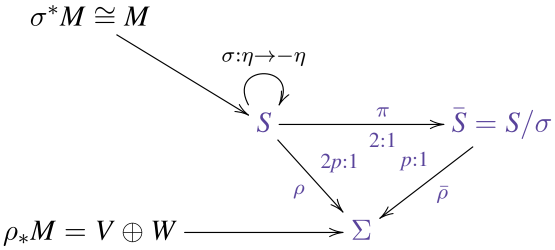
for η tautological section of ρ^*K and $a_i\in H^0(\Sigma,K^{2i})$ ∘
$M:=\operatorname{coker}(\rho^*\Phi-\eta)$ rank 2 vector bundle on S.



Spectral data approach III

Spectral data for $Sp(2p, 2p)$-Higgs bundles (*thesis*)

There is a one to one correspondence between stable $Sp(2p, 2p)$-Higgs bundle $(E = V \oplus W, \Phi)$ on Σ for which $\mathrm{char}(\Phi)^{1/2} = 0$ defines a smooth curve, and the spectral data (S, M) where

(a) the curve $\rho : S \to \Sigma$ is a smooth $2p$-fold cover with equation

$$\eta^{2p} + a_1\eta^{2p-2} + \ldots + a_{p-1}\eta^2 + a_p = 0,$$

in the total space of K, where $a_i \in H^0(\Sigma, K^{2i})$, and η is the tautological section of ρ^*K. The curve S has a natural involution σ acting by $\eta \mapsto -\eta$;

(b) M is a rank 2 vector bundle on S with $\Lambda^2 M \cong \rho^* K^{-2p+1}$, and such that $\sigma^* M \cong M$. Over the fixed points of the involution, the vector bundle M is acted on by σ with eigenvalues $+1$ and -1.

Applications

Connectivity for $\mathcal{M}_{U(p,p)}$

$U(p,p)$-Higgs bundle of fixed degree $\sim (\bar{S}, U_1, D)$ with fixed $\deg M$.

- The choice of D lies in the symmetric product $S^{\tilde{m}}\Sigma$;
- Together with a section s of $K^{2p}[-D]$ with distinct zeros, D gives the map $a_p \in H^0(\Sigma, K^{2p})$;
- The choice of a_p lies in a vector bundle of rank $(4p-1)(g-1)-\tilde{m}$ over $S^{\tilde{m}}\Sigma$, whose total space is $\mathcal{E}$; There is a natural map

$$\alpha : \mathcal{E} \to H^0(\Sigma, K^{2p})$$

- The choice of $\bar{S}$ is given by a point in a Zariski open $\mathcal{A}$ in

$$H^0(\Sigma, K^{2p}) \oplus \bigoplus_{i=1}^{p-1} H^0(\Sigma, K^{2i});$$

- The choice of U_1 is given by a fibration of Jacobians $\mathcal{J}ac$ over $\mathcal{A}$;

Applications

Connectivity for $\mathcal{M}_{U(p,p)}$ (*thesis*)

Each pair of invariants $(m, \tilde{m})$ labels exactly one connected component of $\mathcal{M}_{U(p,p)}$ which intersects the non-singular fibres of the Hitchin fibration

$$\mathcal{M}_{GL(2p,\mathbb{C})} \to \mathcal{A}_{GL(2p,\mathbb{C})}.$$

This component is given by the fibration of $\alpha^*\mathcal{J}ac$ over a Zariski open subset in

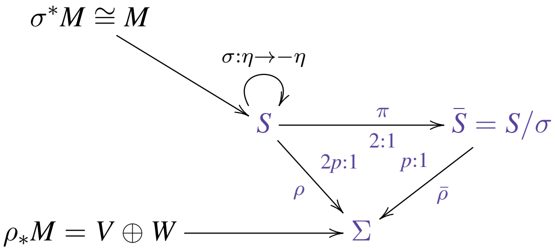
$$\mathcal{E} \oplus \bigoplus_{i=1}^{p-1} H^0(\Sigma, K^{2i}).$$

$\mathcal{M}_{U(p,q)}$ via Morse theory by Bradlow–García-Prada– Gothen '02

Applications

Connectivity for $\mathcal{M}_{Sp(2p,2p)}$

$Sp(2p,2p)$-Higgs bundles with smooth spectral curve $\sim (S, M)$ for M rank 2 vector bundle with $\Lambda^2 M \cong \rho^* K^{-2p+1}$ and conditions on $\sigma^* M \cong M$.



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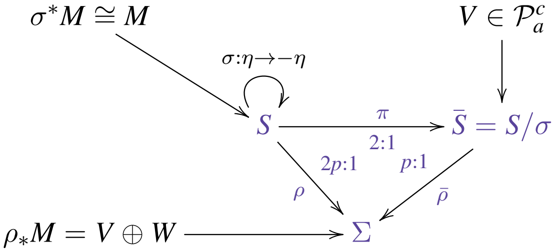

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$Sp(2p,2p)$-Higgs bundles with smooth spectral curve $\sim (S, M)$ for M rank 2 vector bundle with $\Lambda^2 M \cong \rho^* K^{-2p+1}$ and conditions on $\sigma^* M \cong M$.

- $\mathcal{N}^\sigma$ = fixed point set of σ in the moduli space of stable rank 2 vector bundles of determinant $\rho^* K^{2p-1}$;
- $\mathcal{P}_a$ the moduli space parabolic rank 2 vector bundles on $\bar{S}$ whose marked points are the fixed points of the involution σ, whose weights are $1/2$ and whose flag is by the distinguished eigenspaces corresponding to the eigenvalue -1 of σ [Andersen-Grove 2006];
  - Vector bundles in $\mathcal{P}_a$ are stable [Nitsure 86];
  - $\mathcal{P}_a = \mathcal{P}_a^+ \sqcup \mathcal{P}_a^c$ through a natural involution on $\mathcal{P}_a$;
  - $\mathcal{P}_a^c$ is connected [Nitsure 86];

The choice of M is given by an element in

$$\mathcal{N}^\sigma \cong \mathcal{P}_a^c$$

APPLICATIONS

CONNECTIVITY FOR $\mathcal{M}_{Sp(2p,2p)}$ (*thesis*)

The space $\mathcal{M}^s_{Sp(2p,2p)}$ is given by the fibration of $\mathcal{P}^c_a$, over a Zariski open set in the space

$$\bigoplus_{i=1}^{p} H^0(\Sigma, K^{2i}).$$

$\mathcal{M}_{Sp(2p,2q)}$ via Morse Theory by García-Prada–Oliveira '12

Applications

Topological invariants

Milnor-Wood type inequalities for the Toledo invariant $\tau(v,w)$ associated to G-Higgs bundles appear naturally from the spectral data...

- $U(p,p)$-Higgs bundles, for which $\tau(v,w) = v - w$
  - The invariant $\tilde{m} = w - v + 2p(g-1)$ is the number of fixed points of σ with certain property.
  - Fixed points of σ are zeros of $a_p \in H^0(\Sigma, K^{2p})$, thus

$$0 \leq w - v + 2p(g-1) \leq 4p(g-1)$$

$$\boxed{|\tau(v,w)| = |v-w| \leq 2p(g-1)}$$

- $SU(p,p)$-Higgs bundles, for which $\tau(v,w) = v = -w$
  - Methods for $U(p,p)$ can be adapted, and we get

$$\boxed{|\tau(v,w)| = |v| \leq p(g-1)}$$

- Also for $Sp(2n,\mathbb{R})$, $Sp(2p,2p)$... and possibly others?

Thank you for listening!