Special Kähler Manifolds

Wolfgang Globke

School of Mathematical Sciences
Strings Journal Club

THE UNIVERSITY of ADELAIDE

I. Motivation

- Seiberg-Witten theory
- $N = 2$ supergravity
- Moduli spaces of Calabi-Yau 3-manifolds

II. Kähler Manifolds

Definition

Let M be a complex manifold with complex structure J.

1. M is Hermitian if there exists a pseudo-Riemannian metric on M such that
$$\langle JX, JY \rangle = \langle X, Y \rangle$$
for all vector fields X, Y.
2. Then M is a Kähler manifold if J is parallel,
$$DJ = 0,$$
where D is the Levi-Civita connection.

Alternative Definition

The Kähler form ω on a Hermitian manifold M is given by

$$\omega(X, Y) = \langle X, JY \rangle.$$

M is a Kähler manifold $\quad \Leftrightarrow \quad$ ω is closed ($d\omega = 0$)

III. Special Kähler Manifolds

Definition

A Kähler manifold M is called special if there exists a flat torsion-free connection ∇ on M such that

- $(d_\nabla J)(X, Y) \stackrel{\text{def}}{=} (\nabla_X J)Y - (\nabla_Y J)X = 0$
- $\nabla\omega = 0$

Example

If M is a Kähler manifold and D is flat, then setting $\nabla = D$ makes it into a special Kähler manifold, and $\nabla J = 0$ holds.

Conversely, if ∇ is flat and $\nabla J = 0$, then $D = \nabla$ follows.

Special Kähler Domains

A special Kähler domain is a connected open subset $U \subset \mathbb{C}^n$ with a holomorphic function F such that the matrix

$$\mathrm{Im}\left(\frac{\partial^2 F}{\partial z_i \partial z_j}\right)$$

is regular.

- Kähler potential:

$$f = \frac{1}{2}\mathrm{Im}\Big(\sum_{i,j} \frac{\partial F}{\partial z_i}\bar{z}_j\Big)$$

- U is a Kähler manifold with $\omega = \mathrm{i}\partial\bar{\partial} f$ and $\langle\cdot,\cdot\rangle = \omega(\mathrm{i}\cdot,\cdot)$.

Special Flat Coordinates

A special Kähler domain U admits special flat coordinates

$$x_i = \mathrm{Re}(z_i), \quad y_j = \mathrm{Re}\left(\frac{\partial F}{\partial z_i}\right).$$

- Then $\omega = 2\sum \mathrm{d}x_i \wedge \mathrm{d}y_i$.
- Induce flat connection ∇ on U such that $\nabla\omega = 0$.
- U becomes a special Kähler manifold.
- Special Kähler manifolds covered by flat coordinate charts.
- Coordinate changes in $\mathrm{Sp}(2n, \mathbb{R}) \ltimes \mathbb{R}^{2n}$.

IV. Realisations of Special Kähler Manifolds

Definition

Consider $T^*\mathbb{C}^n$ with

- canonical coordinates $(z_1, \ldots, z_n, w_1, \ldots, w_n)$
- symplectic form $\Omega = \sum \mathrm{d}z_i \wedge \mathrm{d}w_i$
- Hermitian form $h = \mathrm{i} \cdot \Omega(\cdot, \tau \cdot)$ of signature (n, n) (τ complex conjugation)

Let M be a complex manifold, $\dim_{\mathbb{C}} M = n$.
A holomorphic immersion $\varphi : M \to T^*\mathbb{C}^n$ is called

- Lagrangian if $\varphi^*\Omega = 0$,
- non-degenerate if $\varphi^* h$ is non-degenerate.

Theorem 1 [ACD]

Lagrangian and non-degenerate φ induces by restricting to M:

- local coordinates $x_i = \mathrm{Re}(z_i)|_M$, $y_i = \mathrm{Re}(w_i)|_M$
- flat torsion-free connection ∇ on M
- $\omega = 2 \sum \mathrm{d}x_i \wedge \mathrm{d}y_i$
- Kähler metric $\langle \cdot, \cdot \rangle = \mathrm{Re}(\varphi^* h)$

Then:

1. M is a special Kähler manifold with ∇ and ω.
2. ω is the Kähler form for $\langle \cdot, \cdot \rangle$.
3. The x_i, y_j yield special flat coordinate charts for M.

Remark

Fact: A holomorphic Lagrangian immersion is locally a closed holomorphic 1-form

$$\varphi_U : U \to T^*\mathbb{C}^n.$$

Assume $\varphi_U = \mathrm{d}F$ for some holomorphic function F (shrink U):

$$w_i = \frac{\partial F}{\partial z_i},$$

as required for special Kähler domains.

Theorem 2 [ACD]

Let M be a simply connected special Kähler manifold, $\dim_{\mathbb{C}} M = n$.

1. There exists a holomorphic non-degenerate Lagrangian immersion $\varphi : M \to T^*\mathbb{C}^n$ inducing $\langle \cdot, \cdot \rangle$, ω and the flat connection ∇.
2. φ is unique up to affine symplectic transformation preserving the canonical real structure of $T^*\mathbb{C}^n$.

a little detour through the harsh realm of affine differential geometry. . .

Definition

Let M be a smooth affine manifold, $\dim M = n$, and
let $\varphi : M \to \mathbb{R}^{n+1}$ be an immersion.

The choice of a transversal vector field ξ on $\varphi(M)$ determines:

- Affine connection ∇ on M.
- Bilinear form $b(\cdot,\cdot)$ given by

$$\overline{\nabla}_X \varphi_*(Y) = \varphi_*(\nabla_X Y) + b(X,Y)\cdot\xi$$

  for vector fields X, Y tangent to M.
- Volume form $\vartheta = \det(\xi,\ldots)$ on M.

Then φ is called an affine immersion.

Definition

If b is non-degenerate, this is independent of the choice of ξ.

In this case there exists a unique transversal field ξ (up to sign) such that

1. $\nabla \vartheta = 0$,
2. ϑ coincides with the volume form induced by b.

φ with this choice of ξ is called a Blaschke immersion.

Definition

The affine shape operator S is defined by

$$\overline{\nabla}_X \xi = SX + \alpha(X)\xi.$$

An affine hypersphere is a Blaschke immersion $\varphi : M \to \mathbb{R}^{n+1}$ with shape operator

$$S = \lambda \cdot \mathrm{id}, \quad \lambda \in \mathbb{R}.$$

It is called

- proper if $\lambda \neq 0$,
- parabolic if $\lambda = 0$.

Examples

Affine hyperspheres:

- proper: sphere
- parabolic: elliptic paraboloid
- parabolic: hyperbolic paraboloid

Fundamental Theorem of Affine Differential Geometry

Let M be a simply connected manifold with torsion-free connection ∇ and non-degenerate metric $\langle\cdot,\cdot\rangle$. Then:

There exists a Blaschke immersion $\varphi : M \to \mathbb{R}^{n+1}$
with induced connection ∇ and Blaschke metric $b = \langle\cdot,\cdot\rangle$.

$$\Leftrightarrow$$

The volume form for $\langle\cdot,\cdot\rangle$ is ∇-parallel and ∇^* is torsion-free and projectively flat.

In the special Kähler case:

- volume form $\sim \omega^m$ is ∇-parallel
- $\nabla^* = J \circ \nabla \circ J$ is torsion-free and flat

Theorem 3 [BC-1]

Let M be a simply connected special Kähler manifold, $\dim_{\mathbb{R}} M = 2n$, with flat connection ∇ and Kähler metric $\langle \cdot, \cdot \rangle$.

Then there exists a Blaschke immersion $\varphi : M \to \mathbb{R}^{2n+1}$ with induced connection ∇ and Blaschke metric $b = \langle \cdot, \cdot \rangle$.

φ is a parabolic hypersphere.

Theorem 4 [BC-1]

A parabolic hypersphere M is a Blaschke immersion of a special Kähler manifold.

$$\Leftrightarrow$$

There exists a complex structure on M such that b is Hermitian and $\omega = b(\cdot, J\cdot)$ is ∇-parallel.

Corollary

Let M be a special Kähler manifold with positive definite metric:
M complete $\quad \Rightarrow \quad D$ flat

- Follows from a theorem due to Calabi and Pogorelov, stating that a parabolic sphere with positive definite metric is flat.
- First proof by Lu using a maximum principle.

V. Projective Special Kähler Manifolds

Definition

A special Kähler domain U is called conic if

1. $\mathbb{C}^\times \cdot U \subseteq U$.
2. F is homogeneous of degree 2
   (that is $F(\lambda z) = \lambda^2 F(z)$).

A conic special Kähler manifold is a special Kähler manifold covered by charts into conic special Kähler domains.

Definition

A projective special Kähler manifold $\overline{M}$ is the orbit space

$$\overline{M} = M/\mathbb{C}^{\times}$$

of a conic special Kähler manifold M.

Lemma

The Kähler metric $\langle \cdot, \cdot \rangle$ on M induces a Kähler metric $(\cdot, \cdot)$ on $\overline{M}$.
Locally:

$$(\pi_* x, \pi_* y)_{\pi(p)} = \frac{\langle x, y \rangle_p}{\langle p, p \rangle_p} - \left| \frac{\langle x, p \rangle_p}{\langle p, p \rangle_p} \right|^2$$

where $x, y \in T_p\mathbb{C}^{n+1}$.

Example

Let $M = \mathbb{C}^{n+1} \setminus \{0\}$ and $F(z) = \mathrm{i} \cdot \sum z_j^2$.
Then:

- $\overline{M} = \mathbb{CP}^n$.
- $(\cdot, \cdot)$ is the Fubini-Study metric.

Recall Hopf fibration:

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \to \mathbb{CP}^n$$

This is generalised by circle bundles over projective special Kähler manifolds.

Definition

Let M be a conic special Kähler manifold with universal cover $\tilde{M}$.
Let $\varphi : \tilde{M} \to T^*\mathbb{C}^n$ be the immersion as in Theorem 2.

The Kähler potential

$$f(p) = \frac{1}{2} h(\varphi(p), \varphi(p))$$

on $\tilde{M}$ induces a Kähler potential f on M.
For a constant $c > 0$ define

$$M_c = \{p \in M \mid f(p) = c\}.$$

Fact: M_c is invariant under the action of $\mathbb{S}^1 \subset \mathbb{C}^\times$.

Definition

The canonical circle bundle on $\overline{M}$ is

$$\mathbb{S}^1 \hookrightarrow M_{1/2} \to \overline{M}.$$

Theorem 5 [BC-2]

Let $\overline{M}$ be a projective special Kähler manifold and $\mathbb{S}^1 \hookrightarrow M_{1/2} \to \overline{M}$ its canonical circle bundle.

Then $M_{1/2}$ has a canonical structure of a proper affine hypersphere whose structure ("Sasakian") determines the projective special Kähler geometry on $\overline{M}$.

Corollary

Let $\overline{U}$ be projective special Kähler domain with complete positive definite metric.

Then $\overline{U} = \mathbb{CP}^n$ and the metric $(\cdot, \cdot)$ is a multiple of the Fubini-Study metric.

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