Aspects of Supermanifolds

Wolfgang Globke

School of Mathematical Sciences
Strings Journal Club

THE UNIVERSITY of ADELAIDE

I. Motivation from Physics: Fermions and Bosons

Elementary particles

- Elementary particle is represented by an element ψ of some Hilbert space H.
- System composed of multiple particles $\psi_1, \ldots, \psi_k$:

$$\psi_1 \otimes \cdots \otimes \psi_k \;\in\; H_1 \otimes \cdots \otimes H_k$$

Fermions vs. Bosons

Every elementary particle is one of the following:

1. Fermion
   - half-integer spin
   - Pauli exclusion principle:
     two fermions cannot occupy identical states
     $\rightsquigarrow$ antisymmetric tensors

$$\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_k \;\in\; \bigwedge^k H$$

   - examples: electrons, protons, neutrons, neutrinos,...
2. Boson
   - integer spin
   - multiple particles can occupy identical states
     $\rightsquigarrow$ symmetric tensors

$$\psi_1 \psi_2 \cdots \psi_k \;\in\; \mathrm{S}^k H$$

   - examples: photons, pions, W- and Z-bosons,...

Standard Model of Particle Physics

Supersymmetry as an attempt to generalise the non-relativistic standard model to a relativistic model:

- Classical SU_6-symmetry is insufficient (Coleman-Mandula Theorem)
- Relativistic theory might require transformations acting on particle spins (fermions $\leftrightarrow$ bosons)
- State space for one particle:

$$H = H_0 \oplus H_1,$$

  where H_0 = bosonic states and H_1 = fermionic states
- System of n particles:

$$\bigoplus_{k=1}^{n} \left(S^k H_0 \otimes \bigwedge^{n-k} H_1 \right)$$

Supersymmetries

Supersymmetries are transformations exchanging bosonic and fermionic states.

- Infinitesimal symmetries first considered in the 1970s.
- Coordinate computations with real and Grassmann variables.
- Later: Introduction of “superspaces” in particle physics: Spaces parameterised by real variables and Grassmann variables.

References

- V.S. Varadarajan:
  Supersymmetry for Mathematicians: An Introduction
  Chapters 1.7 and 1.8
- S. Weinberg:
  The Quantum Theory of Fields III
  Chapter 24

II. Inspired by Grothendieck: Supermanifolds according to Berezin-Leites and Kostant

Definition of "supermanifold" based on principles of algebraic geometry.

Example: Affine algebraic variety V

- V zero set of some polynomial ideal $\mathfrak{I}$
- algebra of functions on V:

$$\mathcal{O}(V) = \mathbb{C}[X_1, \ldots, X_n]/\mathfrak{I}$$

- points $p \in V \quad \overset{1:1}{\longleftrightarrow} \quad$ maximal ideals $\mathfrak{M} = \mathfrak{M}_p$ in $\mathcal{O}(V)$
- $\rightsquigarrow$ geometry of V encoded in the functions on V

Grothendieck Principle: Generalise by

- allowing arbitrary commutative rings
- associate arbitrary prime ideals to "points"

$\rightsquigarrow$ ringed spaces

For supermanifolds we need the following concepts:

- supercommutative rings/algebras
- sheaves

Definition: Supercommutative Algebra

An $\mathbb{R}$-algebra $\mathcal{A}$ is a superalgebra if it is $\mathbb{Z}_2$-graded. This means

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$$

such that $\deg(x) = \varepsilon$ if $x \in \mathcal{A}_\varepsilon$ and

$$\deg(x \cdot y) = \deg(x) + \deg(y) \mod 2$$

for all homogeneous elements $x, y \in \mathcal{A}$.

Furthermore, $\mathcal{A}$ is called supercommutative if

$$x \cdot y = (-1)^{\deg(x)\deg(y)} y \cdot x$$

for all homogeneous elements $x, y \in \mathcal{A}$.

Example (the obvious one)

The exterior algebra $\mathcal{A} = \bigwedge V$ over some vector space V becomes a $\mathbb{Z}_2$-graded algebra when setting

$$\mathcal{A}_0 = \mathbb{R} \oplus \bigwedge^2 V \oplus \bigwedge^4 V \oplus \dots$$
$$\mathcal{A}_1 = V \oplus \bigwedge^3 V \oplus \bigwedge^5 V \oplus \dots$$

It is supercommutative because

$$x \wedge y = -y \wedge x$$

for all $x, y \in V$.
(This leads to the ~~irritating~~ funny fact that the *alternating algebra* in the category of $\mathbb{R}$-algebras becomes the *symmetric algebra* in the category of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebras.)

Definition: Sheaf

Let X be a topological space.

A sheaf $\mathcal{O}$ of (super)commutative rings is a collection of maps

$$\{\ U \mapsto \mathcal{O}(U)\ \}_{U \text{ open in } X},$$

such that $\mathcal{O}(U)$ is a (super)commutative ring satisfying:

1. For all $W \subset U \subset V$ there exist restriction homomorphisms

$$\varrho^U_W : \mathcal{O}(U) \to \mathcal{O}(W)$$

   satisfying $\varrho^U_W \circ \varrho^V_U = \varrho^V_W$ and $\varrho^U_U = \mathrm{id}_U$.
2. If $U = U_1 \cup \ldots \cup U_k$ and $f_i \in \mathcal{O}(U_i)$, then

$$\left[\exists f \in \mathcal{O}(U)\ \forall i : \varrho^U_{U_i}(f) = f_i\right] \ \Leftrightarrow\ \left[\varrho^{U_i}_{U_i \cap U_j}(f_i) = \varrho^{U_j}_{U_i \cap U_j}(f_j)\right]$$

   and f is unique whenever it exists.

$(X, \mathcal{O})$ is called a (super)ringed space.

Remark

Sheaves generalise...

- the algebras of C^∞-functions on the open subsets of a smooth manifold;
- the algebras of regular functions on open subsets of an affine algebraic variety.

Common abuse of language: $\mathcal{O}(U)$ are the "functions on U".

Definition: Morphisms of Ringed Spaces

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be (super)ringed spaces.

A morphism $(\psi, \psi^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of

1. a continuous map $\psi : X \to Y$
2. and a collection ψ^* of "pullback" homomorphisms

$$\{\ \psi^*_W : \mathcal{O}(W) \to \mathcal{O}(\psi^{-1}(W))\ \}_{W \text{ open in } Y}$$

   which commute with the restriction map.
   They must preserve the $\mathbb{Z}_2$-grading in the super category.

Define isomorphisms accordingly.

Remark

If the rings in the sheaves are rings of functions, then the ψ^* are the pullback maps for these functions from W to $\psi^{-1}(W)$.

Example

A superdomain $U^{n|k}$ consists of an open subset $U \subseteq \mathbb{R}^n$ and the sheaf given by

$$\mathcal{O}(W) = C^\infty(W) \otimes \bigwedge \mathbb{R}^k, \qquad W \text{ open in } U.$$

Notation:

$$\begin{aligned} \mathcal{O}(W) &= C^\infty(W)[\vartheta_1, \ldots, \vartheta_k] \\ &= C^\infty(x_1, \ldots, x_n)[\vartheta_1, \ldots, \vartheta_k], \end{aligned}$$

where $x_1, \ldots, x_n$ are coordinates on U and $\vartheta_1, \ldots, \vartheta_k$ are generators of the exterior algebra.

Definition: Supermanifold

Let $|M|$ be a topological space.

A superringed space $M = (|M|, \mathcal{O})$ is a supermanifold of superdimension $n|k$, if there exists a cover of $|M|$ by open sets W such that

$$(W, \mathcal{O}|_W) \;\cong\; U_W^{n|k} \qquad (U_W^{n|k} \text{ some superdomain}).$$

In other words: M is locally isomorphic to $\mathbb{R}^{n|k}$.

Some Properties of Supermanifolds (I)

Given M, the space $|M|$ becomes a (classical) smooth manifold

$$M_{\mathrm{red}} = (|M|, \mathcal{O}/\mathfrak{J}),$$

where

$$\mathfrak{J} = \mathcal{O}_1 + \mathcal{O}_1^2$$

is the sheaf of ideals generated by the odd elements.

Some Properties of Supermanifolds (II)

Associate to $f \in \mathcal{O}(U)$ its value $f(x)$ at $x \in U$:

- The unique $\lambda \in \mathbb{R}$ such that $f - \lambda$ is not invertible in any neighbourhood of x.
- !!! This does not turn f into a classical function on U:

$$\Big[\forall x \in U : f_1(x) = f_2(x)\Big] \quad \not\Rightarrow \quad \Big[f_1 = f_2\Big]$$

  In particular, any $f \in \mathcal{O}(U)_1$ has constant value 0.

The stalk $\mathcal{O}_x$ at $x \in |M|$ is a local ring with maximal ideal $\mathfrak{M}_x = \ker(f \mapsto f(x))$.

Consequence:
A morphism $(\varphi, \varphi^*) : M \to N$ of supermanifolds induces morphisms $\varphi^*_x : \mathcal{O}_{\varphi(x)} \to \mathcal{O}_x$ mapping $\mathfrak{M}_{\varphi(x)}$ to $\mathfrak{M}_x$.

Morphisms in Coordinates

Let $U^{p|q}$ be a superdomain with coordinates x_i, ϑ_j,
and let M be a supermanifold.

Theorem:

- Let $(\psi, \psi^*) : M \to U^{p|q}$ be a morphism. If

$$a_i = \psi^*(x_i) \ (i = 1, \ldots, p), \quad b_i = \psi^*(\vartheta_i) \ (i = 1, \ldots, q),$$

  then $a_i \in \mathcal{O}(M)_0$ and $b_i \in \mathcal{O}(M)_1$.
- Conversely, if $a_i \in \mathcal{O}(M)_0$ and $b_i \in \mathcal{O}(M)_1$ are given, then there exists a unique morphism $(\psi, \psi^*) : M \to U^{p|q}$ such that

$$a_i = \psi^*(x_i) \ (i = 1, \ldots, p), \quad b_i = \psi^*(\vartheta_i) \ (i = 1, \ldots, q).$$

Morphisms in Coordinates

Given $a_i \in \mathcal{O}(M)_0$ and $b_i \in \mathcal{O}(M)_1$, there exists a unique morphism $(\psi, \psi^*) : M \to U^{p|q}$ such that

$$a_i = \psi^*(x_i) \ (i = 1, \dots, p), \quad b_i = \psi^*(\vartheta_i) \ (i = 1, \dots, q).$$

Sketch of Proof:

- *Existence:*
  - $\vartheta_1, \dots, \vartheta_q$ algebraically generate the exterior algebra, so $\psi^*(\vartheta_i)$ can be chosen arbitrarily in $\mathcal{O}(M)_1$
  - $\Rightarrow$ enough to construct homomorphism $C^\infty(U) \to \mathcal{O}(M)_0$ with $x_i \mapsto a_i$
  - $a_i = y_i + \xi_i$, where $\xi_i \in \mathfrak{I}$ is nilpotent and y_i is not
  - for $f \in C^\infty(U)$ define by formal Taylor expansion:

$$\psi^*(f) = f(y_1 + \xi_1, \dots, y_p + \xi_p) = \sum_{\mathbf{k}}^{<\infty} \frac{1}{\mathbf{k}!} (\partial^{\mathbf{k}} f)(y_1, \dots, y_p) \cdot \xi^{\mathbf{k}}$$

    this sum is finite because the ξ_i are nilpotent
- *Uniqueness:* Use approximation of smooth functions by polynomial functions. □

Example

Define morphism $(\psi, \psi^*) : \mathbb{R}^{1|2} \to \mathbb{R}^{1|2}$ by

$$\psi(x) = x$$

and

$$\psi^*(x) = x + \vartheta_1\vartheta_2,$$
$$\psi^*(\vartheta_i) = \vartheta_i.$$

For arbitrary $f \in C^\infty(\mathbb{R}^1)$, we then have

$$\psi^*(f) = f(x) + f'(x)\vartheta_1\vartheta_2.$$

Geometric Intuition?

- *"M is essentially a classical manifold surrounded by a cloud of odd stuff."* (V.S. Varadarajan)
- *"M can be thought of as* M_{red}*, surrounded by a nilpotent fuzz."* (P. Deligne)

But see Varadarajan, Chapter 4.5, for the notion of points provided by the *functor of points*.

References

- P. Deligne et al.:
  Quantum Fields and Strings: A Course for Mathematicians I
  Part 1, Chapter 2
- Yu.I. Manin:
  Gauge Theory and Complex Geometry
  Chapter 4
- V.S. Varadarajan:
  Supersymmetry for Mathematicians: An Introduction
  Chapter 4

III. Examples

examples are hard to find. . .

Supermanifold = Exterior Bundle?

If E is a real vector bundle over the *differentiable* manifold M_0, let $\bigwedge E$ denote the associated exterior bundle.
Then $M_E = (M_0, \Gamma(\bigwedge E))$ is a supermanifold.

Batchelor's Theorem:
Every supermanifold over M_0 is (non-canonically) isomorphic to M_E for some vector bundle E over M_0.

Remark:

- Many more morphisms in the super category than in the differentiable category.
- Batchelor's Theorem does not hold for analytic supermanifolds.

Complex Projective Superspace

Let $\mathbb{P}^n$ be the complex projective n-space.
Define the complex projective superspace $\mathbb{P}^{n|k}$ as follows:

- For V open in $\mathbb{P}^n$, let V' denote its preimage in $\mathbb{C}^{n+1} \setminus \{0\}$.
- Define action of $t \in \mathbb{C}^\times$ on $\mathcal{A}(V') = \mathcal{H}(V')[\vartheta_1, \dots, \vartheta_q]$ by

$$t.\sum_{\mathbf{i}} f_{\mathbf{i}}(z)\vartheta^{\mathbf{i}} = \sum_{\mathbf{i}} t^{-|\mathbf{i}|} f_{\mathbf{i}}(t^{-1}z)\vartheta^{\mathbf{i}}$$

  and set $\mathcal{O}(V) = \mathcal{A}(V')^{\mathbb{C}^\times}$.
- Then $\mathcal{O}(V) \cong \mathcal{H}(X)[\vartheta_1, \dots, \vartheta_q]$ for some affine subspace $X \subset \mathbb{C}^{n+1}$.
- $\mathcal{O}$ is a sheaf of supercommuting $\mathbb{C}$-algebras on X and

$$\mathbb{P}^{n|k} = (\mathbb{P}^n, \mathcal{O}).$$

See also Manin, Chapter 4.3.

Lie Supergroups

A Lie Supergroup G is a supermanifold with a morphism

$$\mu : G \times G \to G$$

such that there exists a unit $e : \mathbb{R}^{0|0} \to G$ and an inverse map $\iota : G \to G$ such that standard diagrams commute.

The linear supergroup $\mathrm{GL}(p|q)$ is an open supersubmanifold of $\mathbb{R}^{p^2+q^2|2pq}$ given as follows:

- Write coordinates in $\mathbb{R}^{p^2+q^2|2pq}$ as follows:

$$\begin{pmatrix} A_0 & B_1 \\ C_1 & D_0 \end{pmatrix},$$

  where the matrices A_0, D_0 contain the even coordinates and B_1, C_1 the odd coordinates.
- $\mathrm{GL}(p|q)$ is the open supersubmanifold defined by $\det(A_0)\det(D_0) \neq 0$.

References

- M. Batchelor:
  The Structure of Supermanifolds
  Trans. Amer. Math. Soc. 253, 1979
- Yu.I. Manin:
  Gauge Theory and Complex Geometry
  Chapter 4.3
- V.S. Varadarajan:
  Supersymmetry for Mathematicians: An Introduction
  Chapter 4

IV. Differential Calculus and Berezin-Integration

Definition: Derivation

A derivation D on an $\mathbb{R}$-superalgebra $\mathcal{A}$ is an $\mathbb{R}$-linear map satisfying

$$D(x \cdot y) = D(x) \cdot y + (-1)^{\deg(x)\deg(D)} x \cdot D(y).$$

Definition: Vector Fields

Let M be a supermanifold. A vector field on M is a derivation of $\mathcal{O}$, that is a family of derivations

$$\{\ D_U : \mathcal{O}(U) \to \mathcal{O}(U)\ \}_{U \text{ open in } |M|}.$$

The derivations of M form the tangent sheaf $\mathcal{T}M$.

In coordinates:

- Extend derivations $D \in C^\infty(U)$ via $D(\vartheta_i) = 0$.
- Define ∂_{ϑ_i} by $\partial_{\vartheta_i}(\vartheta_k) = \delta_{ik}$.
- The derivations

  $$\partial_{x_j}, \partial_{\vartheta_i}$$

  form a module basis for the derivations of $\mathcal{O}(U)$.

Definition: Tangent Vectors and the Differential

Let M be a supermanifold and $p \in |M|$.
A tangent vector at p is a derivation $v : \mathcal{O}_p \to \mathbb{R}$ of the stalk $\mathcal{O}_p$ into $\mathbb{R}$. They form the tangent space T_pM at p.

The differential $\mathrm{d}\psi_p$ of a morphism $(\psi, \psi^*) : M \to N$ of supermanifolds is the morphism

$$\mathrm{d}\psi_p : T_pM \to T_{\psi(p)}N, \quad v \mapsto v \circ \psi_p^*.$$

Vector fields and differentials have properties similar to the classical case (see Varadarajan, Chapter 4.4).

Integration = Differentiation

The integral on $\mathbb{R}[\vartheta_1, \ldots, \vartheta_k]$ is defined by

$$\int \vartheta^{\mathbf{i}} = 0, \qquad \text{if } |\mathbf{i}| < k,$$
$$\int \vartheta_1 \cdots \vartheta_k = 1.$$

Observe that

$$\int = \partial_{\vartheta_k} \cdots \partial_{\vartheta_1}.$$

Definition: Berezin Integral

On a superdomain $U^{n|k}$ the Berezin integral for compactly supported sections $s = \sum_{\mathbf{i}} s_{\mathbf{i}} \vartheta^{\mathbf{i}}$ is defined as

$$\int : \mathcal{O}_{\mathsf{c}}(U) \to \mathbb{R}, \quad s \mapsto \int_U s_{(1,\dots,k)}(x)\, \mathrm{d}^k x.$$

Change of Variables

For a change of variables given locally as

$$\psi(x, \vartheta) = (y, \eta),$$

let

$$\mathsf{J}\psi = \begin{pmatrix} \frac{\partial y}{\partial x} & -\frac{\partial y}{\partial \vartheta} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial \vartheta} \end{pmatrix}.$$

Theorem:

$$\int s = \int \psi^*(s)\mathrm{Ber}(\mathsf{J}\psi)$$

References

- V.S. Varadarajan:
  Supersymmetry for Mathematicians: An Introduction
  Chapters 4.4 and 4.6