Hamiltonian systems

- (M, ω) symplectic manifold, i.e. $\omega \in \Omega^2(M)$ non-degenerate ($\Rightarrow \dim M = 2n$) and $d\omega = 0$

  Example: $M = TQ^*$, $\omega = d\theta$, $\theta = p_i \, dq^i$ symplectic potential.

  Darboux theorem: Any (M, ω) is locally as above in canonical coordinates.

  Note: M compact $\Rightarrow \nexists$ global symplectic potential θ since $\int_M \omega^n = \int_M d(\theta \wedge \omega^{n-1}) = 0$ (Stoke's thm)

- Hamiltonian vector field: $f \in C^\infty(M) \Rightarrow$
  $\iota(X_f)\omega = df$. $\quad X_f \in \mathcal{X}_H(M) \subset \mathcal{X}(M)$

  Note: $\mathcal{L}_{X_f} \omega = 0$

  Example: $\omega = dp_i \wedge dq^i$, $\quad df = \frac{\partial f}{\partial q^i} dq^i + \frac{\partial f}{\partial p_i} dp_i$

  $\Rightarrow X_f = \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i}$

- Poisson bracket: $\{f, g\} := X_f(g) = \omega(X_f, X_g)$
  ($d\omega = 0 \Rightarrow$ Jacobi identity)

  $(C^\infty(M), \{\cdot,\cdot\}) \longrightarrow \mathcal{X}_H(M)$ Lie algebra homomorphism

- Time evolution: For fixed Hamiltonian function
  $H \in C^\infty(M) \Rightarrow \frac{df}{dt} = \{H, f\} = X_H(f) = \omega(X_H, X_f)$

Group action (symmetries)

- G compact Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$

- Fundamental vector field: $\forall \xi \in \mathfrak{g}$
  $\mathfrak{g} \longrightarrow \mathcal{X}(M)$, $\quad \xi \longmapsto \xi^\#(x) = \frac{d}{dt} e^{-t\xi} \cdot x \Big|_{t=0}$

- $G \times M \longrightarrow M$, $\omega \in \Omega^2(M)^G$, i.e. $\mathcal{L}_{\xi^\#}\omega = 0$
  $(g, x) \longmapsto g \cdot x$ $\quad \forall \xi \in \mathfrak{g}$ $\quad (\Rightarrow d(\iota_{\xi^\#}\omega) = 0)$

- $Ad : G \longrightarrow GL(\mathfrak{g})$, $\quad Ad_g(\xi) = \frac{d}{dt} g \cdot e^{t\xi} g^{-1} \Big|_{t=0}$

- Coadjoint action $\quad \langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \longrightarrow \mathbb{R}$

$$\langle Ad_g^* \lambda, \xi \rangle = \langle \lambda, Ad_g \xi \rangle$$

Hamiltonian action

- Moment map $\quad \mu : M \longrightarrow \mathfrak{g}^* \in C^\infty(M, \mathfrak{g}^*)^G$
  - $\mu(g \cdot x) = Ad_{g^{-1}}^* \mu(x)$
  - $\iota_{\xi^\#}\omega = d\langle \mu, \xi \rangle = d\mu^*(\xi)$

where $\mu^* : \mathfrak{g} \longrightarrow C^\infty(M)$ is the comoment map:

$$\mu^*(\xi)(x) := \langle \mu(x), \xi \rangle$$

Note: $\mu^*(\xi)$ is the Hamiltonian function for $\xi^\# = X_{\mu^*(\xi)}$. $\quad$ $\therefore$ $\mu^* : \mathfrak{g} \longrightarrow \mathfrak{X}_H(M) \subset \mathfrak{X}(M)$ is a Lie algebra anti-homomorphism.

- obstruction to the existence of μ is $H^2(\mathfrak{g}, \mathbb{R})$. In particular, if $\mathfrak{g}$ is semisimple, then by Whitehead lemma $H^2(\mathfrak{g}, \mathbb{R}) = 0$.

Namely, $\langle \mu(gx), \xi \rangle = \langle Ad_{g^{-1}}^* \mu(x), \xi \rangle = \langle \mu(x), Ad_{g^{-1}}(\xi) \rangle$

$$\Rightarrow 0 = \frac{d}{dt}\left(\langle \mu(g_t x), \xi \rangle - \langle \mu(x), Ad_{g_t^{-1}}(\xi) \rangle \right)\Big|_{t=0} \quad \text{where } g_t = e^{t\eta}$$

$$= \omega_x(X_\eta(x), X_\xi(x)) - \langle \mu(x), [\xi, \eta] \rangle$$

$$- \{\mu^*(\eta), \mu^*(\xi)\} - \mu^*([\eta, \xi]) = 0$$

γ

- μ is a conserved quantity; Let (M, ω, H, μ) with invariant Hamiltonian ($H(g\cdot x) = H(x)$), then $\mu : M \longrightarrow \mathfrak{g}^*$ is constant along integral curves of the Hamiltonian flow:

$$\begin{aligned}
\frac{d}{dt} \mu(\rho_t)(\xi) &= d\mu(\rho_t) \frac{d}{dt} \rho_t(\xi) \\
&= \omega_{\rho_t}(X_H \circ \rho_t, \xi^\# \circ \rho_t) \\
&= (\iota_{X_H} \omega)_{\rho_t} (\xi^\# \circ \rho_t) \\
&= dH(\rho_t)(\xi^\# \circ \rho_t) \\
&= \frac{d}{ds} H(e^{s\xi} \cdot \rho_t) \Big|_{s=0} \\
&= \frac{d}{ds} H(\rho_t) \Big|_{s=0} = 0
\end{aligned}$$

$\not{\!\times}$

Example: $M = T\mathbb{R}^3 = \{(\bar{x}, \bar{p}) \in \mathbb{R}^3 \times \mathbb{R}^3\}$

$\omega = dx^i \wedge dp_i$, $H \in C^\infty(M)^G$, $G = SO(3)$

$G \times M \longrightarrow M$; $(\bar{x}, \bar{p}) \longmapsto (R\bar{x}, R\bar{p})$, $R \in G$

$\mathfrak{g} \times M \longrightarrow M$; $(\bar{x}, \bar{p}) \longmapsto X_\xi . (\bar{x}, \bar{p}) = (\xi \bar{x}, \xi \bar{p})$

Moment map: $\mu(\bar{x}, \bar{p})(\xi) = \langle \bar{p}, \xi \bar{x} \rangle$ where $\langle \cdot, \cdot \rangle$ is the invariant bilinear form on $\underline{so(3)} \cong \mathbb{R}^3$. Let ϕ denote the isomorphism:

$$\phi : \mathbb{R}^3 \longrightarrow \underline{so(3)}, \quad \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & -\xi_3 & \xi_1 \\ \xi_3 & 0 & -\xi_2 \\ -\xi_1 & \xi_2 & 0 \end{pmatrix}$$

$$\phi(\xi) . \bar{y} = \xi \times \bar{y}$$

$$\Rightarrow \quad \mu(\bar{x}, \bar{p})(\xi) = \langle \bar{p}, \xi \times \bar{x} \rangle = \langle \bar{x} \times \bar{p}, \xi \rangle$$

$$\Rightarrow \quad \mu(\bar{x}, \bar{p}) = \bar{x} \times \bar{p} \in \mathbb{R}^3 \cong \underline{so(3)}$$

conserved angular momentum. $\not{\!\times}$

- Hamiltonian G-space:

(M, ω, μ) where

- M a G-manifold
- $\omega \in \Omega^2(M)^G$ invariant, closed, non-degenerate
- $\mu \in C^\infty(M, \mathfrak{g}^*)^G$ G-equivariant
- $\iota(\xi^\#)\omega = d\langle \mu, \xi \rangle \quad \forall \xi \in \mathfrak{g}$ (i.e. dual to Hamiltonian)

Fundamental example: $M = \mathcal{O} \subset \mathfrak{g}^*$ coadjoint orbit

$\mathcal{O} = \{ Ad^*_{g^{-1}}(\lambda) \mid g \in G \}$

symplectic form: $\omega_\lambda(X_\xi, X_\eta) = \frac{1}{2}\lambda([\xi, \eta])$

$\forall \xi, \eta \in \mathfrak{g}$.

Moment map: $\mu : \mathcal{O} \longrightarrow \mathfrak{g}^*$ identity map.

Thm: Any (M, ω, μ) with transitive G-action is a covering space of an orbit in $\mathfrak{g}^*$.

Non-linear generalization

- $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$: $d\theta^L = -\frac{1}{2}[\theta^L, \theta^L]$, $d\theta^R = \frac{1}{2}[\theta^R, \theta^R]$

  $(\cdot, \cdot)$ invariant, pos-def. inner product on $\mathfrak{g}$. $\Rightarrow \mathfrak{g} \cong \mathfrak{g}^*$.

- $\mu : M \longrightarrow G$, $\iota(\xi^\#)\omega = \mu^*(\theta^L, \xi)$

  unless $\theta^R = \theta^L$ (e.g. G abelian), incompatible with antisymmetry of ω $\Rightarrow$ WRONG!

- Correct definition:

$$\boxed{\iota(\xi^\#)\omega = \frac{1}{2}\mu^*(\theta^L + \theta^R, \xi)}$$

- Problem 1: ω G-invariant incompatible with $d\omega = 0$;

$$\mathcal{L}_{\xi^\#}\omega = \frac{1}{2}\mu^* d(\theta^L + \theta^R, \xi) + \iota(\xi^\#)d\omega$$

- $\chi = \frac{1}{12}(\theta, [\theta, \theta]) \in \Omega^3(G)$

v_ξ fundamental vector field of adjoint action

$$v_\xi = v_\xi^r - v_\xi^\ell \quad ; \quad v_\xi \cdot g = \frac{d}{dt} e^{t\xi} g e^{-t\xi}\Big|_{t=0}$$

$$\begin{cases} \iota(v_\xi)\theta^L = Ad_{g^{-1}}(\xi) - \xi \\ \iota(v_\xi)\theta^R = \xi - Ad_g(\xi) \end{cases}$$

$$\Rightarrow \quad \iota(v_\xi)\chi = \dots = \frac{1}{2}(\theta^L + \theta^R, \xi)$$

$$\Rightarrow \quad 0 = \iota(v_\xi)(d\omega + \mu^*\chi) \Rightarrow \boxed{d\omega = -\mu^*\chi}$$

- Problem 2: ω degenerate;

Let $\xi \in \mathfrak{g}$ be a solution to the equation

$$Ad_{\mu(x)}(\xi) = -\xi \quad \Rightarrow \quad \xi^\# \in \mathfrak{X}(M)$$

At each $x \in M$, we have

$$\begin{aligned} (\theta^L + \theta^R, \xi) &= (\theta^L, \xi) - (\theta^R, Ad_{\mu(x)}(\xi)) \\ &= (\theta^L, \xi) - (Ad_{\mu(x)^{-1}}\theta^R, \xi) \\ &= (\theta^L - \theta^L, \xi) = 0 \end{aligned}$$

$$\Rightarrow \quad \xi^\#(x) \in \operatorname{Ker}\omega_x$$

- Quasi-Hamiltonian G-space: (M, ω, μ)
  - M a G-manifold, $\omega \in \Omega^2(M)^G$ G-invariant
    $\mu \in C^\infty(M, G)^G$ G-equivariant (with adjoint G-action on G).
  - $d\omega = -\mu^* \chi \qquad (\chi \in \Omega^3(G))$
  - $\iota(\xi^\#)\,\omega = \frac{1}{2}\,\mu^*(\theta^L + \theta^R, \xi)$
  - $\ker \omega_x = \{ \xi^\#(x) \mid \xi \in \ker(\mathrm{Ad}_{\mu(x)} + 1) \}$

    ($\Leftrightarrow$ $\ker(\omega_x) \cap \ker(d_x\mu) = \{0\}$, i.e. μ is transversal to $\ker(\omega)$).

Note: $d\omega \neq 0 \Rightarrow C^\infty(M)$ not a Poisson algebra

ω degenerate $\Rightarrow$ M does not have to be even dimensional.

Equivariant cohomology. (Cartan model)

- $\Omega^*_G(M) = \bigoplus_k \Omega^k_G(M)$

  $$\Omega^k_G(M) = \bigoplus_{2l+j=k} \left(S^l \mathfrak{g}^* \otimes \Omega^j(M) \right)^G$$

  i.e. G-equivariant polynomial functions on $\mathfrak{g}$:

  $$\alpha : \mathfrak{g} \longrightarrow \Omega^*(M)$$

  Equivariance means that $\forall \xi \in \mathfrak{g}, g \in G$:

  $$\alpha(\mathrm{Ad}_g(\xi)) = g^* \alpha(\xi)$$

Note: $g = e^{t\xi} \Rightarrow \mathcal{L}_{\xi^\#}\, \alpha(\xi) = 0$

$\therefore$ $\alpha(\xi)$ invariant under $g = e^{t\xi}$, but not under all of G unless G is abelian.

- $(d_G \alpha)(\xi) = d(\alpha(\xi)) - \iota(\xi^{\#})\alpha(\xi)$

$d_G^2 = 0$

- $\Omega_G^0 = \{$ Smooth invariant functions $\alpha : M \to \mathbb{R}\}$

$H_G^0 = \mathbb{R}^k$, $k = \#$ connected components of M/G.

($\because$ closed forms = locally constant invariant functions.)

$\Omega^1_G = \{G\text{-invariant } 1\text{-forms}\}$, G-closed = α closed and horizontal

$\Omega^2_G = \{ \omega_G = \omega + \mu \}$

Thus, $d\omega = 0$ and $\iota(\xi^{\#})\omega = d\langle \mu, \xi \rangle$ for Hamiltonian G-spaces can be written as

$$d_G \omega_G(\xi) = d\omega + d\langle \mu, \xi \rangle - \iota(\xi^{\#})\omega = 0$$

where $\omega_G(\xi) = \omega + \langle \mu, \xi \rangle$.

- Alternatively, we can write

$$d_G\omega(\xi) = d\omega - \iota(\xi^{\#})\omega = -d\langle \mu, \xi \rangle$$
$$= -\mu^* \chi_G(\xi)$$

where $\chi_G \in \Omega^3_G(\mathfrak{g}^*)$ is the equivariant 3-form defined by $\chi_G(\xi) = d\langle \mathbb{1}_{\mathfrak{g}^*}, \xi \rangle$ where $\mathbb{1}_{\mathfrak{g}^*} : \mathfrak{g}^* \to \mathfrak{g}^*$ is the identity map on $\mathfrak{g}^*$.

- Now, the axioms for quasi-Hamiltonian G-space; $d\omega = -\mu^* \chi$ and $\iota(\xi^{\#})\omega = \frac{1}{2}\mu^*(\theta^L + \theta^R, \xi)$ can be formulated as

$$d_G \omega(\xi) = -\mu^* \chi_G(\xi) \quad , \quad \text{where}$$

$\chi_G \in \Omega^3_G(G)$ is the d_G- closed equivariant 3-form on G defined by

$$\chi_G(\xi) = \chi + \frac{1}{2}(\theta^R + \theta^L, \xi).$$

- $(\omega, \chi_G) \in \Omega^3_G(\mu)$ defines a relative cocycle; in fact a closed G-equivariant class in $H^3_G(\mu, \mathbb{R})$
- Recall the construction of algebraic mapping cone:

$$f: X \to Y, \qquad f^*: C^*(Y) \to C^*(X)$$

$$C^p(f) = C^{p-1}(X) \oplus C^p(Y)$$

$$d_f(\beta, \alpha) := (d\beta + f^*\alpha, -d\alpha)$$

$$\Rightarrow \cdots \to H^p(Y) \to H^p(X) \to H^{p+1}(f) \to H^{p+1}(Y) \to \cdots$$

$$\therefore H^*(f) = H^*(C(f), d_f).$$

- So, a Hamiltonian G-space determines $[\omega_G] \in H^2_G(M, \mathbb{R})$, and a quasi-Hamiltonian G-space fixes $[(\omega, \chi_G)] \in H^3_G(\mu, \mathbb{R})$.
- Pre-quantization. (M, ω, μ) pre-quantizable iff $[\omega_G]$ has an integral lift, i.e. has a preimage under $c_{\mathbb{R}}: H^2_G(M, \mathbb{Z}) \to H^2_G(M, \mathbb{R})$.
  - A q-Hamiltonian space (M, ω, μ) is pre-quantizable iff $[(\omega, \chi_G)] \in H^3_G(\mu, \mathbb{R})$ has an integral lift.

Note: - $\exists$ alternative description in terms of relative gerbes or quasi-line bundles.

- If $\pi_1(G) = 0$, it suffices to require integral lift in non-equiv. relative cohom. $H^3(\mu, \mathbb{R})$.

Basic examples

- Conjugacy classes: $\mathcal{C} \subset G$ (counterpart to $\mathcal{O} \subset \mathfrak{g}^*$, see $\mathcal{O} \subset L\mathfrak{g}^*$ later!)

  $\exists !\ \omega \in \Omega^2(\mathcal{C})^G$ s.t. $\mathcal{C}$ is a q-Hamiltonian G-space, with $\mu : \mathcal{C} \hookrightarrow G$ embedding. Namely, $\forall f \in \mathcal{C}$, $\xi, \eta \in \mathfrak{g}$, we have

$$\omega_f(\xi^\#(f), \eta^\#(f)) = \frac{1}{2}\left[(\eta, \mathrm{Ad}_f \xi) - (\xi, \mathrm{Ad}_f \eta)\right]$$

- Double: $D(G) = G \times G$ (analogue of TG^*)

  G acts diagonally by conjugation.

$$\omega = \frac{1}{2}\left[(pr_1^* \theta^L, pr_2^* \theta^R) + (pr_1^* \theta^R, pr_2^* \theta^L) - \right.$$
$$\left. - (m^* \theta^L, (m \circ T)^* \theta^R)\right]$$

where $m : G \times G \to G$ multiplication,
$T : G \times G \to G \times G$ switch map.

Moment map: $\mu(a,b) = aba^{-1}b^{-1}$ (commutator map.)

$D(G)$ basic building block of moduli space of flat G-bundles on a closed oriented surface Σ; $\mathcal{M}_G(\Sigma) = \dfrac{\mathrm{Hom}(\pi_1(G), G)}{G}$

$$\cong \mu^{-1}(1) /\!/ G$$

- <u>Thm</u> (M_i, ω_i, μ_i) $i=1,2$ q-Hamiltonian G-spaces. $M_1 \times M_2$ with diagonal G-action.

$$\omega = pr_1^* \omega_1 + pr_2^* \omega_2 + \frac{1}{2}\left(pr_1^* \mu_1^* \theta^L, pr_2^* \mu_2^* \theta^R\right)$$

$\mu = m \circ (\mu_1 \times \mu_2)$ defines a q-Hamiltonian G-space $(M_1 \circledast M_2, \omega, \mu)$ called <u>fusion</u>.

- $\circledast$ is associative and commutative (up to Isom.)

- <u>Morphism</u> of q-Hamiltonian G-spaces

$\varphi_G : G \longrightarrow G'$ homomorphism

$\varphi_M : M \longrightarrow M'$ smooth

s.t. $d\varphi_G : \mathfrak{g} \longrightarrow \mathfrak{g}'$ preserves inner products, and

$\varphi_M^* \omega' = \omega$, $\mu \circ \varphi_M = \varphi_G \circ \mu$,

$\varphi_M(g \cdot x) = \varphi_G(g) \cdot \varphi_M(x)$

- <u>Exponentiation</u>

$$\begin{array}{ll} (M, \sigma, \Phi) & \text{Hamiltonian G-space} \\ \big\downarrow \text{exp} & \\ (M, \omega, \mu) & \text{q-Hamiltonian G-space, } \underline{\text{if}} \end{array}$$

$d_\xi \exp$ is bijective $\forall \xi \in \Phi(M)$
(i.e. bijection on $\mathrm{Im}(\Phi)$)

where

$\mu = \exp(\Phi)$, $\omega = \sigma + \Phi^* \bar{\omega}$, where

$$\bar{\omega} = \frac{1}{2} \int_0^1 \left(\exp_s^* \theta^R, \frac{\partial}{\partial s} \exp_s^* \theta^R\right) ds \,; \quad \exp_s(\eta) = \exp(s\eta)$$

- $\bar{\omega}$ satisfies $d\bar{\omega} = -\exp^* \chi$,

$\iota(\xi^\#)\,\bar{\omega} = -d(\cdot,\xi) + \frac{1}{2}\exp^*(\theta^R + \theta^L, \xi)$

Conversely, given a $\mathfrak{g}$-Hamiltonian G-space (M, ω, μ) we can take the logarithm:

$$\Big\downarrow \log$$

$(M, \omega - \mu^* \log^* \bar{\omega}, \log(\mu))$ is a Hamiltonian G-space if exp is a bijection on a neighborhood $\mathcal{U} \subset \mathfrak{g} \longrightarrow V \subset G$.

Hamiltonian loop group action

Motivation: There is a 1-1 correspondence between quasi-Hamiltonian G-spaces and Ham. LG-spaces with proper moment map.

- $LG = C^\infty(S^1, G)$, $L\mathfrak{g} = \Omega^0(S^1, \mathfrak{g}) = H^q(S^1, \mathfrak{g})$

$L\mathfrak{g}^* = \Omega^1(S^1, \mathfrak{g}) = H^{q-1}(S^1, \mathfrak{g})$ smooth dual

$\langle \cdot, \cdot \rangle : L\mathfrak{g}^* \times L\mathfrak{g} \longrightarrow \mathbb{R}$ pairing:

$\langle A, \xi \rangle := \int_{S^1} (A, \xi)_{\mathfrak{g}}$

$\Rightarrow$ $L\mathfrak{g}^* \subset (L\mathfrak{g})^*$ topological dual.

- Coadjoint action $L\mathfrak{g}^* \times \{k\} \subset \widehat{L\mathfrak{g}}^*$

$A^g = \mathrm{Ad}_g A - k\, g^* \theta^R$, $\Omega G \rtimes G = LG$

ΩG acts freely on $L\mathfrak{g}^*$. [level $k=1 \Rightarrow$ usual gauge action]

- $\mathrm{Hol} : L\mathfrak{g}^* \xrightarrow{\Omega G} G$, $\mathrm{Hol} = \mathrm{Hol}_1$ where

$$\mathrm{Hol}_s : L\mathfrak{g}^* \longrightarrow G \ ; \quad \begin{cases} \mathrm{Hol}_s(A)^{-1} \frac{\partial}{\partial s} \mathrm{Hol}_s(A) = A \\ \mathrm{Hol}_0(A) = e \end{cases}$$

$$\mathrm{Hol}(\xi\, ds) = \exp(s\xi)$$

$$\mathrm{Hol}_s(A^g) = g(s)\, \mathrm{Hol}_s(A)\, g(s)^{-1}$$

- $\chi \in \Omega^3(G)$, $\mathrm{Hol}^* \chi = -d\bar{\omega}$ where

$$\bar{\omega} = \frac{1}{2} \int_0^1 \left(\mathrm{Hol}_s^* \theta^R , \frac{\partial}{\partial s} \mathrm{Hol}_s^* \theta^R \right) ds \ , \text{ is}$$

LG-invariant, and

$$\iota(\xi^\#)\, \bar{\omega} = -d \int_{S^1} (A, \xi) + \frac{1}{2} \mathrm{Hol}^* (\theta^L + \theta^R, \xi(0))$$

- Def: A Hamiltonian LG-space (N, σ, ϕ)
  - N Banach manifold w/ LG-action (smooth in appropriate sense)
  - $\sigma \in \Omega^2(M)^{LG}$ invariant
  - $\phi \in C^\infty(N, L\mathfrak{g}^*)^{LG}$ equivariant

such that

+ $d\sigma = 0$
+ $\iota(\xi^\#)\sigma = d \int_{S^1} (\phi, \xi)$
+ σ is weakly non-degenerate, i.e. the induced map $\sigma_x^\flat : T_x N \longrightarrow T_x^* N$ is injective, but no necessarily surjective.

- <u>1-1 correspondence</u>:

Assume Φ is proper, i.e. $\Phi^{-1}(\text{compact}) \subset N$ is compact.

Then,

(1) since $\Omega G \circlearrowright Lg^*$ is free, $\Omega G \circlearrowright N$ will be free. $\Rightarrow M = Hol(N) = N/\Omega G$ called the <u>holonomy manifold</u>.

(2) since Φ is proper, M will be a smooth finite-dimensional manifold.

(3) Residual G-action by $Hol(A)$ on $M \Rightarrow$

$$\begin{tikzcd} LG \times N \arrow[r] \arrow[d] & N \arrow[d] \\ G \times Hol(N) \arrow[r] & Hol(N) \end{tikzcd}$$

and induced G-equivariant map, $Hol(\Phi)$;

$$\begin{tikzcd} N \arrow[r, "\Phi"] \arrow[d, "Hol"'] & Lg^* \arrow[d, "Hol"] \\ Hol(N) \arrow[r, "Hol(\Phi)"] & G \end{tikzcd}$$

Similar to exponentiation, we have

$$\begin{tikzcd} (N, \sigma, \Phi) \arrow[d, "Hol"] & \text{Hamiltonian } LG\text{-space} \\ (Hol(N), \omega, \mu) & q\text{-Hamiltonian } G\text{-space} \end{tikzcd}$$

(with induced G-action as above!)

where

- $\omega = \sigma + \phi^* \bar{\omega}$ (LG-invariant $\Rightarrow$ ω is basic w.r.t. projection $\mathrm{Hol}: N \to \mathrm{Hol}(N)$)

- $\mu = \mathrm{Hol}(\phi)$

<u>Conversely</u>, given a q-Ham. G-space (M, ω, μ) set

$$\begin{tikzcd} N = \mu^* Lg^* \arrow[r, "\phi"] \arrow[d] & Lg^* \arrow[d, "\Omega G"] \\ M \arrow[r, "\mu"] & G \end{tikzcd}$$

$\exists!$ action on N s.t. ϕ is equivariant and the diagram commutes, namely

$$LG \curvearrowright M \times Lg^* ;$$

$$g : (x, A) \longmapsto (g(0).x,\ A^g)$$

and

$$\sigma = \mathrm{Hol}^* \omega - \phi^* \bar{\omega} .$$

<u>Fundamental example</u>: $Lg^* \supset \mathcal{O}$ coadjoint orbits $= N$, with LG-action $\mathcal{O} = LG \cdot (A, \lambda)$. and Kirillov-Kostant-Souriau symplectic structure ω.

All such orbits are pre images $\mathcal{O} = \mathrm{Hol}^{-1}(\mathcal{C})$ of conjugacy classes $\mathcal{C} \subset G$.

$\therefore$ Holonomy manifold of $\mathcal{O}$ is $\mathcal{C}$;

$$\begin{tikzcd} \mathcal{O} \arrow[r, hook] \arrow[d, "\mathrm{Hol}"] & Lg^* \arrow[d, "\mathrm{Hol}"] \\ \mathcal{C} \arrow[r, hook] & G \end{tikzcd}$$

- quantization of above orbit PER of LG and $Q(M_1 \circledast M_2) = Q(M_1) \boxtimes Q(M_2)$

Next: quantization & symplectic reduction.